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Master thesis

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Tautological cycles on Jacobian varieties

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Contents

Acknowledgement

In 2006 I followed a course in abelian varieties given by Gerard van der Geer and Ben Moonen. Subsequently I followed part of the lectures in a *Spring School on Abelian Varieties*, also organized by Van Der Geer and Moonen. The spring school was concluded with a workshop, and one of the speakers there was Arnaud Beauville. The title of his lecture was "Algebraic cycles on Jacobian varieties" and in it he explained what his paper [?] was about. The construction of the Chow ring and algebraic equivalence ring were appealing to me, because I thought they were a kind of algebraic analagons for (co)homology rings. These I had studied for my bachelor thesis about Poincaré duality. At the time I was already thinking about my master thesis and whom to write it with. My enthusiasm for abelian varieties, and my curiosity about the Chow ring brought me to Gerard van der Geer, who works as a professor at the Universiteit van Amsterdam. He agreed to become my advisor. Because I was registered as a student at the Universiteit Utrecht, I needed an advisor in Utrecht too. Prof. dr. Gunther Cornellissen was so kind to fill in this role. I would like to thank them both for the time they have spent on this thesis.
Despite the fact that I was curious and enthusiastic, there were quite some gaps in my knowledge on algebraic and complex geometry. For instance, before I started this thesis, I didn't know much about the Hodge decomposition theorem, let alone about the deformations of complex manifolds or the variation of the Abel-Jacobi mappings in a family. A few times I visited Frans Oort for some help. He always took the time, and explained things very clearly. I would like to thank him for his time and advise. I would also like to thank Alberto Collino, who mailed me a copy of his article [?]. At the time I needed it, there was no copy of this article publicly available in the Netherlands.
I would also like to thank my friends at the Universiteit Utrecht. Not only for their moral support, but also for listening to my mathematical troubles. I'd like to mention Charlene Kalle and Ruden Teuben in particular. I have really enjoyed every single cup of coffee we've drunk toghether, every hour we've studied together.
Finally, I'd like to thank my parents, sister and brother-in-law for their moral support and all the faith they have had in me during my studies.

Preface

In this thesis I present several results concerning the Chow ring and the algebraic equivalence ring of a Jacobian variety. The most important reason for studying these objects is that they carry information that the (co)homology rings do not carry. The algebraic equivalence ring contains a subring, the tautological ring, and for different curves, these rings are really different. One reason for this is the absence of a Poincaré formula in the algebraic equivalence ring of a generic curve. This absence is a corollary of Ceresa's theorem. This in turn makes Ceresa's theorem into the cornerstone of this thesis. The proof that Ceresa gave is of an analytic nature, and it took quite some time for me to grasp all of its ingredients. I have tried to write down the proof as clearly as I could, in a somewhat more general context than Ceresa did. A first advantage was that I could simplify a (small) step in the proof. A second is that it made it easier to compare his proof to other proofs. We will see that Harris' theory of harmonic volumes doesn't give a new proof (for part of the theorem), by reasoning on the dimensions of the involved moduli spaces. This is overcome by taking a closer look to the involved mappings (the normal functions): Collino and Pirola consider the infinitesimal invariants to obtain a new proof of part of Ceresa's theorem.
A lot of the objects that I work with (Chow ring, algebraic equivalence ring, Beauville decompostion, (primitive) intermediate Jacobians, Hodge theory) were new or rather vague to me. Fortunately, there are many references for these subjects, and I have tried to be as adequate as possible in providing them.

The thesis itself is organized as follows

In *section 1* we introduce the Chow ring and the algebraic equivalence ring of an abelian variety. We show that is possesses a second gradation besides the one by (co)dimension. This gives the Beauville decomposition, which is exploited in section 4.

In *section 2* we recall the definition of a Jacobian variety. Our first approach is of an algebraic nature, our second analytic. The latter comes from a more general construction: the intermediate Jacobian. These in turn have primitive parts, and they are of our interest in view of section 3. We are particularly interested in how they vary in a family.

In *section 3* we reap the fruits of our work in section 2. We can now state and prove the theorem of Ceresa. Moreover, our rather general considerations in section 2 allow us to compare the proof by Ceresa to other proofs. That is, we discuss the harmonic volumes by Harris and we'll see why this doesn't generalize to higher genera. We also sketch a proof of part of Ceresa's theorem given by Collino and Pirola.

In *section 4* we compare the homology ring of a Jacobian variety to its algebraic equivalence ring. In particular, we compare their tautological subrings. In cohomology, this ring is rather dull: it has the same structure for every curve (of fixed genus). This is due to the Poincaré formula, which is absent in the algebraic context. This absence is due to Ceresa's theorem, and it inspires to investigate the tautological ring. We give generators for this ring and describe relations between them.

1 The Chow ring

In this chapter we introduce the Chow ring $\mathrm{CH}(X)$ of a non-singular algebraic variety X. It is a graded commutative ring that we can associate to such a variety. We may think of it as a generalization of the divisor class group of a nonsingular variety. Its elements are formal sums of subvarieties of X, divided out by an equivalence relation. The ring structure is given by the intersection product, the graduation by codimension.
The rest of the chapter is devoted to Chow rings of abelian varieties. We will see that they carry a second product, the Pontryagin product. Furthermore, they carry a second grading. This gives rise to the Beauville decomposition of $\mathrm{CH}(X)$.
We close the chapter by introducing the algebraic equivalence ring of a variety. This is a quotient of the Chow ring. In particular, we will see that the algebraic equivalence ring of an abelian variety inherits the two gradations of the Chow ring.

1.1 Algebraic cycles

We start with looking at divisors on a nonsingular variety X. A prime divisor on X is a closed integral subscheme Y of codimension one. A Weil divisor is an element of the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors. A divisor is written as a formal sum

$$D = \sum n_i Y_i,$$

where the n_i are integers and only finitely many of them are different from zero. Let Y be a prime divisor on X and let η be its generic point. The local ring $\mathscr{O}_{\eta,Y}$ is a discrete valuation ring with quotient field $K(X)$, the function field of X. The corresponding valuation is denoted with

$$\nu_Y : K(X)^* \to \mathbb{Z}.$$

An element $r \in K(X)$ is said to have a zero along Y if $\nu_Y(r) > 0$ and a pole if $\nu_Y(r) < 0$. The order of a zero or pole is $|\nu_Y(r)|$. For fixed $r \in K(X)^*$, all but finitely many valuations ν_Y are zero in r.
This allows us to define the divisor of a function $r \in K(X)^*$ by $\mathrm{div}(r) = \sum \nu_Y(r) \cdot Y$, where the sum is taken over all prime divisors on X. Such a divisor is called a principal divisor. Taking the principal divisor of a function gives an injective homomorphism of groups $K(X)^* \to \mathrm{Div}(X)$ since $\mathrm{div}(rs) = \mathrm{div}(r) + \mathrm{div}(s)$ and $\mathrm{div}(1) = 0$. The cokernel of this map is called the divisor class group of X and is denoted by $\mathrm{Cl}(X)$.

We will now generalize this idea. First of all, we want to define divisors and principal divisors on a variety that is not necessarily nonsingular. But in that case we do not have discrete valuations at our disposal, since the local rings need not be regular. Second, we want to define formal sums of closed subvarieties of higher codimension. We want to divide this group by an equivalence relation that extends the linear equivalence relation of divisors.

Let X be a variety and let Y be a closed subvariety of codimension one. The free abelian group generated by the closed subvarieties of X of codimension one is denoted by $\mathscr{Z}^1(X)$. An element of $\mathscr{Z}^1(X)$ is called an algebraic cycle of codimension one and it is written as a formal sum

$$D = \sum n_i \cdot Y_i,$$

where the integers n_i are zero, except for finitely many.
Let Y be a closed subvariety of X of codimension one. Let $\eta \in X$ be the generic

point of Y and let $\mathscr{O}_{\eta,Y}$ be the local ring of Y. For a function $r \in K(X)^*$ we define the order of vanishing of r along Y by

$$\mathrm{ord}_Y(r) := \ell_{\mathscr{O}_{\eta,X}}(\mathscr{O}_{\eta,X}/(r)),$$

where $\ell_{\mathscr{O}_{\eta,X}}(\mathscr{O}_{\eta,X}/(r))$ is the length of $\mathscr{O}_{\eta,X}/(r)$ as an $\mathscr{O}_{\eta,X}$-module. For fixed $r \in K(X)^*$, the value $\mathrm{ord}_Y(r)$ is zero except for finitely many closed subvarieties Y of X. This allows us to define an algebraic cycle of codimension one for a given $r \in K(X)^*$ by

$$\mathrm{div}(r) := \sum \mathrm{ord}_Y(r) \cdot Y,$$

where the sum is taken over all the closed subvarieties Y of X of codimension one. The notation is the same as above, which is justified by the fact that in case of a nonsingular variety, the divisor of a function and the associated codimension 1-cycle are the same. Furthermore, taking the corresponding algebraic cycle of a function on X gives a homomorphism of groups

$$K(X)^* \to \mathscr{Z}^1(X),$$

since $\mathrm{div}(rs) = \mathrm{div}(r) + \mathrm{div}(s)$ and $\mathrm{div}(1) = 0$. The image $\mathrm{div}(K(X))$ is denoted by $\mathscr{Z}^1(X)_{rat}$. The cokernel of this map is called the codimension-one Chow group, denoted with $\mathrm{CH}^1(X)$. It is defined for any variety and if X is nonsingular, it coincides with $\mathrm{Div}(X)$. The proofs of these statements can be found in [?], App. A.

Next we generalize the notion of a divisor with respect to the dimension. Let X be a variety of dimension n. The free abelian group of closed subvarieties of codimension k is denoted by $\mathscr{Z}^k(X)$. Its elements are called algebraic cycles of codimension k and they are written as finite formal sums

$$\sum n_i Y_i,$$

where the n_i are integers and the Y_i are closed subvarieties.

Example 1.1 A natural situation in which algebraic cycles appear is the following. Let $Z \subset X$ be a closed subscheme of finite type, all of whose irreducible components are of dimension k in X. The geometric multiplicity of a component Z_j is defined by $m_j = \ell_{\mathscr{O}_{Z_j,Z}}(\mathscr{O}_{Z_j,Z})$. Let $Z_{j,red}$ be the reduced scheme associated to Z_j. Then $Z_{j,red}$ can be viewed as a k-dimensional subvariety of X. To Z we associate an algebraic k-cycle defined by $\sum m_j Z_{j,red}$. By abuse of notation we denote this cycle by Z.

Sometimes it is convenient to emphasize the dimension of an algebraic cycle, rather than the codimension. In that case we write $\mathscr{Z}_k(X)$. Thus the group of codimension k-cycles $\mathscr{Z}^k(X)$ is the same as the group of $(n-k)$-cycles $\mathscr{Z}_{n-k}(X)$.

Next, we look at the functorial behavior of associating the group of cycles to a variety. Let $f : X \to X'$ be a proper morphism of varieties and Y a closed subvariety of dimension k of X. Then $f(Y) = Y'$ is a closed subvariety of dimension $k' \leq k$ of X'. If $k < k'$ we define the k-cycle f_*Y to be the zero cycle in $\mathscr{Z}_k(X)$. If $k' = k$, then $K(Y')$ is a finite extension of $K(Y)$ of some degree, say d. In this case we define $f_*Y = d \cdot Y' \in \mathscr{Z}^k(X)$. We can extend this map $\mathbb{Z}$-linearly to obtain a group homomorphism

$$f_* : \mathscr{Z}_k(X) \to \mathscr{Z}_k(X').$$

Recall that a morphism $f : X \to X'$ of varieties is called flat if for every point $x \in X$, the stalk $\mathscr{O}_{f(x),Y}$ is a flat $\mathscr{O}_x$-module via the natural map $f^\sharp : \mathscr{O}_{f(x),Y} \to f_*\mathscr{O}_{x,X}$. It follows from the definition that a flat morphism $f : X \to X'$ of relative dimension n associates to a closed k-dimensional subvariety Y' of X' a $(k+n)$-cycle $f^{-1}(Y')$, denoted with $f^*(Y')$. We extend this map $\mathbb{Z}$-linearly to a homomorphism of groups

$$f^* : \mathscr{Z}_k(X') \to \mathscr{Z}_{k+n}(X).$$

The maps f_* and f^* are called the push-forward and the pull-back by f. If f is not proper, the push-forward is not defined, since the image of a closed subvariety might not be closed. In contrast to this, we will see that we can define the pull-back of cycles by maps that are not flat.

1.2 The Chow ring

We have seen that an element $r \in K(X)^*$ of the function field of a variety of dimension n gives rise to the $(n-1)$-cycle $\mathrm{div}(r)$, the divisor of r. The free group of divisors on X that is generated by the divisors of elements r in $K(X)^*$ is denoted with $\mathscr{Z}^1(X)_{rat}$. This is a subgroup of $\mathscr{Z}^1(X)$ and we say that two divisors D and D' are rationally equivalent if their difference $D - D'$ is in $\mathscr{Z}^1(X)_{rat}$. In this subsection we generalize this equivalence relation to higher codimension.

Let X be an n-dimensional variety and Y a $(k+1)$-dimensional closed subvariety of X. An element $r \in K(Y)^*$ gives rise to a divisor $\mathrm{div}(r)$ on Y. Since the inclusion $Y \subset X$ is proper, we can push this divisor forward to obtain a k-cycle in $\mathscr{Z}_k(X)$. By abuse of notation, this cycle is also written as $\mathrm{div}(r)$. An element $Z \in \mathscr{Z}_k(X)$ is rationally equivalent to zero, written $Z \sim_{rat} 0$, if there are finitely many $(k+1)$-dimensional subvarieties Y_i of X and $r_i \in K(Y_i)^*$ such that $Z = \sum \mathrm{div}(r_i)$. Note that $-Z = \sum \mathrm{div}(r_i^{-1})$. It follows that the set of k-cycles that are rational equivalent to zero, denoted with $\mathscr{Z}_k(X)_{rat}$, is a subgroup of $\mathscr{Z}_k(X)$. The quotient

$$\mathrm{CH}_k(X) := \mathscr{Z}_k(X)/\mathscr{Z}_k(X)_{rat}$$

is called the k-dimensional Chow group. Usually we want to emphasize the codimension of the cycles. In this case we write $\mathrm{CH}^{n-k}(X)$ instead of $\mathrm{CH}_k(X)$. The class of a subvariety $Y \subset X$ in the Chow group is denoted by $[Y] \in \mathrm{CH}_k(X)$
The Chow groups behave nicely under push-forward by a proper- and pull-back by a flat morphism, which is due to the following proposition.

PROPOSITION 1.2 Let $f : X \to X'$ be a morphism of varieties.

a If f is proper then $f_*(\mathscr{Z}_k(X)_{rat}) \subset \mathscr{Z}_k(X')_{rat}$.

b If f is flat of relative dimension n then $f^*(\mathscr{Z}_k(X')_{rat}) \subset \mathscr{Z}_{k+n}(X)_{rat}$

PROOF. See e.g. Thm.1.4 and Thm.1.7 of [**?**] or Lemma 9.5 and 9.7 of [**?**]. □

Note that $\mathrm{CH}^k(X) = 0$ for $k < 0$ and $k > n$. The (total) Chow group of X is defined by

$$\mathrm{CH}(X) := \sum_{k=1}^{n} \mathrm{CH}^k(X).$$

The Chow group of a nonsingular variety is equipped with a product that makes it a commutative ring. Before we define it, we look at a simple case first. Recall (e.g. from [**?**] or [**?**]) that on a projective nonsingular surface X there exists a unique pairing $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ such that (1) for two curves C, D in X that intersect properly[1], the pairing $C \cdot D$ counts the number of points in the intersection $C \cap D$,

[1] Two subvarieties C, D intersect properly if all the irreducible components have codimension $\mathrm{codim}(C) + \mathrm{codim}(D)$.

(2) it is symmetric and additive and (3) it depends on the linear equivalence classes: if $C' \sim_{rat} C$ then $C \cdot D = C' \cdot D$. The proof runs in two steps. First, it is shown that for any two curves C, D, there is a curve $C' \sim_{rat} C$ such that C' and D intersect properly. Second, it is shown that there exists a pairing for curves that intersect properly and that it satisfies the above requirements. The pairing $C \cdot D$ is then defined by the pairing $C' \cdot D$ of properly intersecting curves, with $C' \sim_{rat} C$. The proof is concluded by showing uniqueness of the pairing.
This is *not* an intersection product on the Chow ring, but it gives rise to one. We expect the intersection of two curves to consist of the points of the intersection, together with the right multiplicity. This pairing gives back only an integer: the degree of the 0-cycle that consists of the points in the intersection. For an intersection product we want to get more information out of the pairing, i.e. we want to know the rational equivalence classes of the points in the intersection, together with the right multiplicities.
Instead of move one curve in its rational equivalence class and then count the points in the intersection, we can calculate the pairing in another way: by its intersection multiplicities. Let C, D be two curves without a common irreducible component. Given a point $x \in C \cap D$ we define $i(C, D; x)$ by the length of the $\mathscr{O}_{x,X}$-module $\mathscr{O}_{x,X}/(c, d)$, where c and d are the local equations of C and D at x. It can be shown that for such curves, we have $C \cdot D = \sum_{x \in C \cap D} i(C, D; x)$. It remains to mention a method to calculate the self-intersection of a curve C. This can be done by the degree of the normal sheaf of C in X.

We now define an intersection product on the Chow ring of a projective surface. The class of the surface itself is the identity. The intersection of two points or a point and a curve is always zero. The intersection of two curves without a common irreducible component is defined by the 0-cycle

$$C \cdot D = \sum_{x \in C \cap D} i(C, D; x)[x].$$

It remains to show which 0-cycle is attached to $C \cdot C$. This is a technical matter that we mention without motivation. Let $i : C \to X$ be the inclusion and let $\mathscr{N}_{C/X}$ the the normal sheaf on C. This is an invertible sheaf and its corresponding divisor is denoted by $c_1(\mathscr{N}_{C/X})$. Pushing it forward to X gives an expression for the self-intersection of C, namely $C \cdot C = i_*(c_1(\mathscr{N}_{C/X}))$.

This intersection product can be successfully generalized to higher dimensions. It gives the Chow ring the structure of a commutative ring. Furthermore, it has some nice functorial properties that we will exploit later on. We formulate this in a theorem.

Theorem 1.3 Let $\mathfrak{C}$ be the category of nonsingular quasi-projective varieties over $\mathbb{C}$. Then there is a unique intersection theory for the cycles modulo rational equivalence. By an intersection theory we mean a pairing $\mathrm{CH}^k(X) \times \mathrm{CH}^m(X) \to \mathrm{CH}^{k+m}(X)$ for each X in $\mathfrak{C}$, that is subject to the following axioms.

A1 The intersection pairing makes $\mathrm{CH}(X)$ into a commutative associative graded ring, with identity, for every X in $\mathfrak{C}$. It is called the *Chow ring* of X.

A2 For any morphism $f : X \to Y$ of varieties in $\mathfrak{C}$, the map $f^* : \mathrm{CH}(Y) \to \mathrm{CH}(X)$ is a homomorphism of groups. If $g : Y \to Z$ is another morphism in $\mathfrak{C}$, we have $f^* \circ g^* = (f \circ g)^*$.

A3 For any proper morphism $f : X \to Y$ of varieties in $\mathfrak{C}$, $f_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ is a homomorphism of graded groups (which shifts degrees). If $g : Y \to Z$ is another proper map of varieties in $\mathfrak{C}$, we have $g_* \circ f_* = (g \circ f)_*$.

A4 (*Projection formula*). If $f : X \to Y$ is a proper morphism, $x \in \mathrm{CH}(X)$ and $y \in \mathrm{CH}(Y)$, then
$$f_*(x \cdot f^*y) = f_*(x) \cdot y.$$

A5 (*Reduction to the diagonal*). If $\alpha, \beta \in \mathrm{CH}(X)$, and if $\Delta : X \to X \times X$ is the diagonal morphism, then
$$\alpha \cdot \beta = \Delta^*(\alpha \times \beta).$$

A6 (*Local nature*). If Y and Z are subvarieties of X which *intersect properly* (which means that every irreducible component of $V \cap W$ has codimension equal to $\mathrm{codim}(Y) + \mathrm{codim}(Z)$), we can write
$$[Y] \cdot [Z] = \sum i(Y, Z; W_j)W_j,$$
where the sum runs over the irreducible components W_j of $Y \cap Z$, and where the integer $i(Y, Z; W_j)$ depends only on a generic point of W_j on X. We call $i(Y, Z; W_j)$ the *local intersection multiplicity* of Y and Z along W_j.

A7 (*Normalization*). If V is a subvariety of X, and D is an effective Cartier divisor meeting V properly, then $V \cdot D$ is the cycle $V \cap D$ on V, which is defined by restricting the local equation of D to V. (This implies in particular that transversal intersections of non-singular subvarieties have multiplicity 1).

Here, the pull-back f^* is defined for every map $f : X \to Y$ in $\mathfrak{C}$ by
$$f^*(\alpha) = p_{X*}(\Gamma_f \cdot p_Y^{-1}(\alpha))$$
for an element $\alpha \in \mathrm{CH}^k(Y)$. The variety Γ_f denotes the graph of f in $X \times Y$ and p_X, p_Y are projections from the product onto its factors.

Proof. We briefly sketch two proofs. The first one is classical, and references for it can be found in [?]. The second proof is described in the first eight chapters of [?].
The classical proof consists of three parts. First one shows that the theory, if it exists, is unique. Second is Chow's moving lemma. Two subvarieties Y, Z of X are said to intersect properly if all the irreducible components are of codimension $\mathrm{codim}(Y) + \mathrm{codim}(Z)$. The lemma states that for every pair of subvarieties Y, Z, there is a subvariety Y', rationally equivalent to Y, that has a proper intersection with Z. Third, we need an intersection multiplicity. Let Y, Z be subvarieties of X intersecting properly. Let W be an irreducible component, with local ring $\mathscr{O}$. The varieties Y and Z are, locally at W, given by ideals $\mathfrak{y}$ and $\mathfrak{z}$. The intersection multiplicity that does the job is defined by
$$i(Y, Z; W) = \sum_{i \geq 0} \ell(\mathrm{Tor}_i^{\mathscr{O}}(\mathscr{O}/\mathfrak{y}, \mathscr{O}/\mathfrak{z})).$$

Fulton's intersection theory is rather complicated and too involved to explain here. The main difference is that it avoids Chow's moving lemma. Instead, it assigns to every pair of cycles Z and Z' a cycle class in the Chow ring of the intersection scheme of the supports of the cycles. This gives a class in $\mathrm{CH}(X)$ and it doesn't change if we vary Z or Z' in their rational equivalence classes. In the various chapters of his book [?] Fulton develops this theory and shows that his product satisfies the axioms A1-A7. □

1.3 Beauville decomposition

As we mentioned before, we are interested in the Chow rings of the Jacobian variety of curves. A first step in understanding this ring better, is to study the Chow ring of an abelian variety, because the Jacobian of a nonsingular curve is an abelian variety. Arnaud Beauville showed that the Chow ring of an abelian variety carries a second gradation (besides gradation by codimension). We introduce it by giving an interpretation of Mumford's formula for divisor classes.

First we recall this formula. Let A be an abelian variety over $\mathbb{C}$ of dimension g. An integer $n \in \mathbb{Z}$ gives an isogeny $[n] : A \to A$. The pull-back is a homomorphism $[n]^* : \mathscr{Z}^1(A) \to \mathscr{Z}^1(A)$ and Mumford's formula gives an expression for $[n]^*(D)$ in the Chow group $\mathrm{CH}^1(A)$ by

$$[n]^* D \sim_{rat} \left(\frac{n^2+n}{2}\right) D + \left(\frac{n^2-n}{2}\right) [-1]^* D.$$

If we extend scalars by $\mathrm{CH}^1_{\mathbb{Q}}(A) := \mathrm{CH}^1(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, we can write

$$[D] = \frac{1}{2}([D] + [-1]^*[D]) + \frac{1}{2}([D] - [-1]^*[D])$$

for any divisor class $[D] \in \mathrm{CH}^1_{\mathbb{Q}}(A)$. So every divisor class can be written as the sum of a symmetric and an anti-symmetric divisor class[1]. Let $\mathrm{CH}^1_0(A) \subset \mathrm{CH}^1_{\mathbb{Q}}(A)$ denote the subgroup of symmetric divisor classes and $\mathrm{CH}^1_1(A) \subset \mathrm{CH}^1_{\mathbb{Q}}(A)$ the subgroup of anti-symmetric divisor classes. Since a (non-trivial) symmetric divisor class is not anti-symmetric in $\mathrm{CH}^1_{\mathbb{Q}}(A)$ and vice versa[2], we have

$$\mathrm{CH}^1_{\mathbb{Q}}(A) = \mathrm{CH}^1_0(A) \oplus \mathrm{CH}^1_1(A).$$

Mumford's formula gives another interpretation of this decomposition: the group $\mathrm{CH}^1_0(A) \subset \mathrm{CH}^1_{\mathbb{Q}}(A)$ is the group consisting of divisor classes $[D]$ such that $[n]^*[D] = n^2 \cdot [D]$ and $\mathrm{CH}^1_0(A) \subset \mathrm{CH}^1_{\mathbb{Q}}(A)$ is the subgroup consisting of divisor classes $[D]$ such that $[n]^*([D]) = n \cdot [D]$.

We conclude that the endomorphisms $[n]^*$ on the codimension one Chow groups have eigenvalues of the form n^{2-s}, with $s = 0, 1$, and these two types of eigenvalues give rise to a decomposition of the Chow group $\mathrm{CH}^1_{\mathbb{Q}}(A)$.

This result can be generalized to higher codimensions. Let $\mathrm{CH}^k_{\mathbb{Q}}(A)$ denote the Chow group with extended scalars $\mathrm{CH}^k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. In this way we lose the torsion of the Chow groups, which is crucial for the theorem.

Theorem 1.4 (A. Beauville, 1986) For $s \in \mathbb{Z}$, let $\mathrm{CH}^k_s(A)$ denote the subgroup of $\mathrm{CH}^k_{\mathbb{Q}}(A)$ consisting of classes α such that $[n]^*(\alpha) = n^{2k-s} \cdot \alpha$ for every $n \in \mathbb{Z}$. Then

$$\mathrm{CH}^k_{\mathbb{Q}}(A) = \bigoplus_{s=k-g}^{k} \mathrm{CH}^k_s(A).$$

Before we can give the proof, we have to introduce two concepts: the Pontryagin product and the Fourier transform.

The multiplication on A is denoted by $\mu : A \times A \to A$, the identity by $e : \mathrm{Spec}(\mathbb{C}) \to A$. This structure on A gives rise to a second product in the Chow ring $\mathrm{CH}(A)$ called

[1]A divisor class $[D]$ is called symmetric/anti-symmetric if $[-1]^*[D] = \pm[D]$

[2]An element that is both symmetric as anti-symmetric is a torsion point. But there are none since we tensored with $\mathbb{Q}$

the Pontryagin product. Let p_i denote the projection $A \times A \to A$ onto its i-th factor. Given two elements $\alpha \in \mathrm{CH}^k(A)$ and $\beta \in \mathrm{CH}^m(A)$, we define

$$\alpha \star \beta = \mu_*(p_1^*(\alpha) \cdot p_2^*(\beta)) \in CH^{k+m-g}(A).$$

This is the Pontryagin product, and it is a commutative and associative product on the Chow ring.
Let A^t denote the dual abelian variety of A and let $\wp \in \mathrm{CH}^1(A \times A^t)$ denote the divisor class of the Poincaré bundle $\mathscr{P}$ on $A \times A^t$. We use the formal expression $\exp(\wp)$ to denote the sum

$$1 + \wp + \frac{1}{2!}\wp \cdot \wp + \frac{1}{3!}\wp \cdot \wp \cdot \wp + \cdots .$$

Note that this is a finite sum, since the k-fold self-intersection of a divisor is zero if $k > \dim(A)$. The Mukai-Fourier transform $\mathscr{F} : \mathrm{CH}_{\mathbb{Q}}(A) \to \mathrm{CH}_{\mathbb{Q}}(A^t)$ is defined by

$$\mathcal{F}(\alpha) = p_{A^t *}(\exp(\wp) \cdot p_A^* \alpha),$$

where p_A and p_{A^t} are the projections of $A \times A^t$ onto their factors and $\mathrm{CH}_{\mathbb{Q}}(A)$ is the Chow ring with extended scalars. Shigeru Mukai introduced this transform in a different context (as a functor) in [?]. Arnaud Beauville studied in [?] the particular case that we consider here. From now on we shall refer to $\mathcal{F}$ simply as the Fourier transform.

Lemma 1.5 The Fourier transform $\mathcal{F} : \mathrm{CH}_{\mathbb{Q}}(A) \to \mathrm{CH}_{\mathbb{Q}}(A^t)$ has the following properties:

- F1 Let $\mathcal{F}^t : \mathrm{CH}_{\mathbb{Q}}(A^t) \to \mathrm{CH}_{\mathbb{Q}}(A)$ be the Fourier transform of A^t, where we have identified $A^{tt} = A$. Then $\mathcal{F}^t \circ \mathcal{F} = (-1)^g[-1]^*$.
- F2 $\mathcal{F}(\alpha \star \beta) = \mathcal{F}(\alpha) \cdot \mathcal{F}(\beta)$ and $\mathcal{F}(\alpha \cdot \beta) = (-1)^g \mathcal{F}(\alpha) \star \mathcal{F}(\beta)$.
- F3 Let $f : A \to A$ be an isogeny. Then $\mathcal{F} \circ f^* = f_*^t \circ \mathcal{F}$ and $\mathcal{F} \circ f_* = f^{t*} \circ \mathcal{F}$.
- F4 Let $\alpha \in \mathrm{CH}^p_{\mathbb{Q}}(A)$ and write $\mathcal{F}(\alpha) = \sum_q \beta^q$, with $\beta^q \in \mathrm{CH}^q_{\mathbb{Q}}(A^t)$. Then $[k]^*(\beta_q) = k^{g-p+q} \cdot \beta^q$.

Proof. This is Prop. 3 of [?]. □

Remark 1.6 In classical analysis, the Fourier transform provides a isometry

$$L^2(V) \to L^2(V^*),$$

where V is a real vector space and V^* its dual. The Fourier transform of the convolution product of two functions equals the normal product of the transforms of the functions. In view of F2, the map $\mathcal{F}$ is called the Fourier transform.

Before we give a proof of the Beauville decomposition we make some observations for an element $\alpha \in \mathrm{CH}^p_{\mathbb{Q}}(A)$.
(1) Application of A6 of theorem 1.3 on the map $[k] : A \to A$ and the cycles $1 \in \mathrm{CH}^0_{\mathbb{Q}}(A)$ and $\alpha \in \mathrm{CH}^p_{\mathbb{Q}}(A)$ yields

$$[k]_*[k]^*(\alpha) = [k]_*(1) \cdot \alpha = k^{2g} \cdot \alpha.$$

The second equality follows from the definition of $[k]_*$ and the degree of $[k]$ being k^{2g}. It follows that $[k]_*[k]^* = k^{2g} \cdot \mathrm{id}_{CH(A)}$. If $\alpha \in \mathrm{CH}^p_s(A)$, then $[k]_*[k]^*(\alpha) = k^{2p-s} \cdot k_*(\alpha) = k^{2g} \cdot \alpha$; we conclude

$$\mathrm{CH}^p_s(A) = \{\alpha \in \mathrm{CH}_{\mathbb{Q}}(A) : \quad [k]_*(\alpha) = k^{2g-2p+s} \cdot \alpha, \text{ for all } k \in \mathbb{Z}\}.$$

(2) If $\mathcal{F}(\alpha) \in \mathrm{CH}^{g-p+s}_{\mathbb{Q}}(A^t)$, then $\mathcal{F}(\alpha) \in \mathrm{CH}^{g-k+s}_s(A^t)$. Indeed, application of F4, with $\beta^{g-p+s} = \mathcal{F}(\alpha)$ yields $[k]^*(\beta_{g-p+s}) = k^{2g-2p+s} \cdot \beta^{g-p+s}$.
(3) Furthermore, $\mathcal{F}(\alpha) \in \mathrm{CH}^s_{\mathbb{Q}}(A^t)$ if and only if $\alpha \in \mathrm{CH}^p_s(A)$. To see this, put $\beta = (-1)^g[-1]^*\mathcal{F}(\alpha)$ and $q = g - p + s$. Then $\beta \in \mathrm{CH}^q_{\mathbb{Q}}(A^t)$ and $\mathcal{F}^t(\beta) = \alpha \in \mathrm{CH}^{g-q+s}_{\mathbb{Q}}(A)$. Our previous observation now implies that $\alpha \in \mathrm{CH}^p_s(A)$. For the converse, we have $[k]_*[k]^* = k^{2g} \cdot \mathrm{id}_{CH(A)}$. So $[k]_*(\alpha) = k^{2g-2p+s} \cdot \alpha$ and by F3 we find

$$[k]^*\mathcal{F}(\alpha) = \mathcal{F}([k]_*(\alpha)) = k^{2g-2p+s} \cdot \alpha.$$

By F4 we conclude that $\mathcal{F}(\alpha)$ is of dimension $p - s$.

Proof of theorem 1.4. Let $\alpha \in \mathrm{CH}^p_{\mathbb{Q}}(A)$ and put $\beta = (-1)^g[-1]^*\mathcal{F}(\alpha)$. The latter decomposes as $\sum_q \beta^q$ with $\beta^q \in \mathrm{CH}^q_{\mathbb{Q}}(A^t)$. From the observations we just made, it follows that $\beta^q \in \mathrm{CH}^q_{q+p-g}(A^t)$ and thus $\mathcal{F}^t(\beta^q) \in \mathrm{CH}^p_{p+q-g}(A)$. We conclude that $\alpha = \sum_q \mathcal{F}^t(\beta^q)$. We only need to show that the decomposition is unique. But this follows from the fact that a non-trivial element in $\mathrm{CH}^p_m(A) \cap \mathrm{CH}^p_n(A)$ is a torsion element, while $\mathrm{CH}_{\mathbb{Q}}(A)$ doesn't have torsion. □

According to the theorem, every Chow group decomposes into g summands. The following proposition shows that several of these summands vanish.

Proposition 1.7 (1) If $s < 0$ and $p \in \{0, 1, g-1, g\}$ then $\mathrm{CH}^p_s(A) = 0$. The same space is also trivial if $p \leq g - 2$ and $s < p - g + 2$.

Proof: This is Prop. 3 of [?]. □

So in the case of $g = 5$ and $p = 2$ we might have non-trivial $\mathrm{CH}^2_{-1}(A)$. In general, it remains an open question whether the spaces $\mathrm{CH}^p_s(A)$ with $s < 0$ are trivial.

1.4 Algebraic equivalence

On an abelian variety we can translate subvarieties using the group law. In particular on a Jacobian variety, where certain subvarieties carry information about the curve. But a translation of such a subvariety will carry the same information. We will make this more precise later, but for now this is the reason for looking at a quotient of the Chow ring of an abelian variety: the algebraic equivalence ring.

Before we introduce the subgroup of cycles that are algebraically equivalent to zero, we give a geometric definition of a cycle being rationally equivalent to zero. To this end we consider a variety X and the product $X \times \mathbb{P}^1$ with $p_X, p_{\mathbb{P}^1}$ the projections onto its factors. Let $Z = \sum Z_i$ be a $(k+1)$-cycle on $X \times \mathbb{P}^1$ such that every for all i, the map $f_i := p_{\mathbb{P}^1}|_{Z_i} : Z_i \to \mathbb{P}^1$ is dominant. The subscheme $f^{-1}(a)$ can be viewed a cycle on Z_i and thus on X. As a cycle on X it is denoted with $Z_i(a)$. In this way, the cycle Z gives a family of cycles $Z(a) = \sum Z_i(a)$ on X with $a \in \mathbb{P}^1$. It can be shown (Prop. 1.6 of [?]) that a cycle on X of the form $Z(0) - Z(\infty)$ is rationally equivalent to zero and conversely, any cycle in $\mathscr{Z}^k(X)_{rat}$ is of the form $Z(0) - Z(\infty)$, with Z a $(k+1)$-cycle on $X \times \mathbb{P}^1$ satisfying the above condition.

We use this description to define algebraic equivalence. The only difference is that $\mathbb{P}^1$ may be replaced by any nonsingular variety T. To be precise:

Definition 1.8 Let X and T be nonsingular varieties of dimensions g, m respectively. Let $Z = \sum_i Z_i$ be a $(k+m)$-cycle such that the varieties Z_i satisfy the following condition: the map $f_i : Z_i \to T$, which is the restriction of $p_T : X \times T \to T$, is dominant. For $t \in T$, we view the subscheme $f_i^{-1}(t)$ of Z_i as a k-cycle on X. A cycle a of the form $Z(t_0) - Z(t_1)$ is called algebraically equivalent to zero, written as $a \sim_{alg} 0$. Two cycles Z and Z' are algebraically equivalent if their difference is algebraically equivalent to zero.

If two cycles V and W are algebraically equivalent to zero, then so is their sum $V+W$. Furthermore, the algebraic equivalence relation is maintained under proper push-forward and flat pull-back. A closer look to the definition reveals that the parameter space T in the definition may be replaced by a nonsingular projective curve. Finally, two positive k-cycles V, W are algebraically equivalent on X, i.e. $V \sim_{alg} W$, if and only if there is a nonsingular, projective curve T, a positive cycle U on X and a positive cycle Z together with points t_0 and t_1 such that $V + U = Z(t_0)$ and $W + U = Z(t_1)$. In other words, two positive cycles that are algebraically equivalent cannot necessarily be deformed into one another via some algebraic curve: we might need to add a positive cycle to it first. For proofs of these statements we refer to Chap. 10 of [**?**].

The group of k cycles that are rationally equivalent to zero is denoted with $\mathscr{Z}_k(A)_{alg}$ or $\mathscr{Z}^{g-k}(A)_{alg}$ if we want to stress the codimension. We have an inclusion of groups

$$\mathscr{Z}^k(A) \supset \mathscr{Z}^k(A)_{alg} \supset \mathscr{Z}^k(A)_{rat}.$$

The quotient $\mathscr{Z}^k(A)/\mathscr{Z}^k(A)_{alg}$ is denoted with $\mathcal{A}^k(A)$. The direct sum

$$\mathcal{A}(A) := \bigoplus_{i=0}^{g} \mathcal{A}^i(A)$$

is the same as the quotient of $\mathrm{CH}(A)$ and $\bigoplus(\mathscr{Z}^k(A)_{alg}/\mathscr{Z}^k(A)_{rat})$. The latter is an ideal of the Chow ring and it follows that $\mathcal{A}(A)$ is a ring. It is called the algebraic equivalence ring of A. Just as before, we denote $\mathcal{A}^k_{\mathbb{Q}}(A)$ and $\mathcal{A}_{\mathbb{Q}}(A)$ for the group and the ring with extended scalars.

The ideal $\bigoplus(\mathscr{Z}^k(A)_{alg}/\mathscr{Z}^k(A)_{rat}) \subset \mathcal{A}_{\mathbb{Q}}(A)$ is also homogeneous with respect to the Beauville decomposition. Indeed, consider

$$\alpha = \alpha_{(k-g)} + \alpha_{(k-g+1)} + \cdots + \alpha_{(k)} \in \mathscr{Z}^k(A)_{alg}/\mathscr{Z}^k(A)_{rat}$$

and the elements $[n]_*\alpha$ for $n = 1, \ldots, g+1$. These elements are also algebraically equivalent to zero and the linear space that they generate lies in the span of $\{a_{(k-g)}, \ldots, \alpha_{(k)}\}$. A base change is given by a nonsingular integral matrix[1], so each $\alpha_{(j)}$ is algebraically equivalent to zero. This shows in particular that $\mathcal{A}_{\mathbb{Q}}(A)$ inherits both gradations.

Remark 1.9 We close this section by noting that in view of the gradations on $\mathcal{A}_{\mathbb{Q}}(A)$, the intersection and Pontryagin products are homogeneous of some degree. More precise, for elements $\alpha \in \mathcal{A}^p_s(A)$ and $\beta \in \mathcal{A}^q_t(A)$ we have

$$\alpha \cdot \beta \in \mathcal{A}^{p+q}_{s+t}(A) \quad \text{and} \quad \alpha \star \beta \in \mathcal{A}^{p+q-g}_{s+t}(A).$$

[1]The n-th column is of the form $n^{p-g} \cdot (1, n, n^2, \ldots, n^g)$. So the matrix is a Vandermonde matrix whose n-th column is multiplied by n^{p-g}.

2 Jacobian Varieties

In the previous section we defined the algebraic equivalence ring for a variety and we have seen that for an abelian variety, it carries two gradations. In this section we turn our attention to Jacobian varieties and their algebraic equivalence rings. First we recall some facts about the Jacobian variety. Then we look at the classes of certain cycles in the Chow ring of a Jacobian variety, the Ceresa cycles. In the last two subsections we make preperations for section 3. In particular, we recall some facts about intermediate Jacobians and normal functions.

In this section, a curve is a complete, nonsingular connected curve over $\mathbb{C}$ of some genus $g \neq 0$. This holds for this section only.

2.1 Jacobian varieties

Let C be a curve and let $D = \sum n_i P_i$ be a divisor on C, where $P_i \in C$ and n_i an integer. The associated invertible sheaf is denoted with $\mathscr{O}(D)$. Recall that the map $D \mapsto \mathscr{O}(D)$ is a 1-1 correspondence between $\mathrm{Div}(C)$ and the invertible sheaves on C and that $\mathscr{O}(D_1 - D_2) \cong \mathscr{O}(D_1) \otimes \mathscr{O}(D_2)^{-1}$. Furthermore, two divisors are rationally equivalent if and only if their associated invertible sheaves are isomorphic. The quotient $\mathrm{Div}(C)/\sim_{rat}$ is denoted with $\mathrm{Cl}(C)$ and is called the divisor class group. We could also have written $\mathrm{CH}^1(C)$ for this group, but this is not customary.
The homomorphism $\mathrm{Cl}(C) \to \mathrm{Pic}(C)$ is an isomorphism of groups. This allows us to define the degree of the class of an invertible sheaf: it is just the degree of the associated divisor. This is well defined, because if two divisors both represent the same class of invertible sheaves, then they differ by a rational divisor, which is of degree zero. Let $\mathrm{Pic}^0(C)$ denote the group of classes of invertible sheaves of degree zero. Loosely speaking, the Jacobian variety is an abelian variety over $\mathbb{C}$ whose closed points are isomorphic to the group $\mathrm{Pic}^0(C)$.
We will now define it rigorously, as the variety that represents the functor of a moduli problem. Let T be a connected scheme over $\mathbb{C}$ and let $\mathscr{L}$ be an invertible sheaf on $C \times T$. The projections of $C \times T$ onto its factors are denoted by p_C and p_T. For any point $t \in T$ we can consider the restriction $\mathscr{L}|_{C \times \{t\}}$ as an invertible sheaf on C. We denote it by $\mathscr{L}_t$ and it can be shown that its degree is independent of t. If $\mathscr{M}$ is any invertible sheaf on T then p_T^* is an invertible sheaf on $C \times T$ all of whose restrictions $(p_T^* \mathscr{M})_t$ are trivial. In particular, they are of degree zero. To a scheme T over $\mathbb{C}$ we can associate the group

$$P_C^0(T) = \{\mathscr{L} \in \mathrm{Pic}(C \times T) |\, \deg(\mathscr{L}_t) = 0 \text{ for all } t\}/p_T^* \mathrm{Pic}(T).$$

This association is a functor from the category of schemes over $\mathbb{C}$ to the category of abelian groups.

Theorem 2.1 There is an abelian variety $J(C)$ over $\mathbb{C}$ and a morphism of functors $\iota : P_C^0 \to J$ such that $\iota(T) : P_C^0(T) \to J(C)(T)$ is an isomorphism whenever $C(T)$ is non-empty.

Proof. See e.g. section 4 of [**?**]. □

In our case there is always a rational point, since we work over the complex numbers. So we can view the curve C as a pointed variety: a variety with a specified rational point. Likewise, $(J(C), o)$ is a pointed variety, where $o \in J(C)(\mathbb{C})$ is the identity. This leads to the following definition. For two pointed varieties (X, x) and (Y, y), a divisorial correspondence between them is an invertible sheaf $\mathscr{L}$ on $X \times Y$ such that the restriction to both $X \times \{y\}$ and $\{x\} \times Y$ are trivial.

THEOREM 2.2 Let P be a $\mathbb{C}$-rational point on C. Then there is a divisorial correspondence $\mathscr{M}^P$ between (C, P) and $(J(C), o)$ with the following universal property: for any pointed variety (T, t) and any divisorial correspondence $\mathscr{L}$ between (C, P) and (T, t) there is a unique morphism $\phi : T \to J(C)$ such that $(1 \times \phi)^* \mathscr{M}^P \cong \mathscr{L}$.

PROOF. See e.g. Thm. 1.2 of [?]. □

This has an interesting consequence. Consider the divisorial correspondence between (C, P) and itself via the sheaf $\mathscr{L}^P := \mathscr{O}(\Delta - \{P\} \times C - C \times \{P\})$ on $C \times C$, where Δ is the diagonal. It has the property that $\mathscr{L}^P|_{C \times \{Q\}} \cong \mathscr{O}(P - Q)$. By theorem 2.2 there is a unique map $\phi^P : C \to J(C)$ with $\phi^P(P) = o$ and $(1 \times f^P)^* \mathscr{M}^P = \mathscr{L}^P$. If we identify $J(C)(\mathbb{C})$ with $\mathrm{Cl}^0(C)$, the divisors on C of degree zero, then ϕ^p can be regarded as the map sending Q to $[Q - P]$. This map is in fact a closed immersion. For injectivity, look at the fibers of ϕ^P: these are $\mathbb{P}^1$'s or $\mathbb{P}^0$'s. If $\mathbb{P}^1$ occurs as a fiber, then $C = \mathbb{P}^1$, contradicting with the genus $g \geq 1$. The rest of the proof can be found in [?].

The subvarieties W_r. The image of ϕ^P is denoted by W_1 and it is the subvariety that paramatrizes divisor classes of degree zero that can be represented by $Q - P$, for $Q \in C$. Note that the image of $\phi^{P'}$ is a translation of W_1 by the point $[P - P']$. All the translates of W_1 carry the same information: they paramatrize the divisor classes that can be represented by an effective divisor of degree one.
We generalize this to higher dimensions. Let $r \geq 1$ be an integer. The closed points of $C^r = C \times C \times \cdots \times C$ represent effective divisors on C of degree r. Let Σ_r be the symmetric group on r letters and consider its action on C^r via

$$\sigma : (Q_1, \ldots, Q_r) \mapsto (Q_{\sigma(1)}, \ldots, Q_{\sigma(r)}).$$

The quotient of C^r by this action is denoted by $C^{(r)}$ and it is called the r-th symmetric power of C.

PROPOSITION 2.3 The symmetric power $C^{(r)}$ is a nonsingular variety. It has the following universal property: every symmetric morphism[1] $C^r \to T$ factors through $\pi : C^r \to C^{(r)}$.

PROOF. See e.g. §3 of [?]. □

Consider the morphism $\phi_r^P : C^r \to J(C) : (Q_1, \ldots, Q_r) \mapsto \sum_i \phi^P(Q_i)$. It is clearly symmetric so it factors through the symmetric product. We get a proper morphism

$$\phi_{(r)}^P : C^{(r)} \to J(C) : Q_1 + \cdots + Q_r \mapsto \sum_i \phi^P(Q_i).$$

The image is a subvariety of $J(C)$ that paramatrizes the divisor classes that can be represented by an effective divisor of degree r. On the closed points this map is $\sum_i Q_i \mapsto \sum [Q_i - P]$. The fiber of $\phi_{(r)}^P$ over a point $\sum[Q_i - P]$ is the linear system $|\sum Q_i|$. The variety W_r paramatrizes the divisor classes that can be represented by a positive divisor of degree r. This is independent of the embedding of C in $J(C)$. Any other choice of $P \in C$ would give a translate of the W_r, but it still paramatrizes the same kinds of linear systems. In other words:

The varieties W_r carry geometric information: they paramatrize linear systems on the curve of the form $|D|$, with D positive of degree r. This information is independent of the choice of embedding $\phi^P : C \to J(C)$.

[1] A morphism $\varphi : C^r \to T$ is called symmetric if $\varphi \circ \sigma = \varphi$, for every $\sigma \in \Sigma_r$.

If $r \geq g$ then $W_r = J(C)$. Indeed, let $[D]$ be a divisor class of degree zero. Then $[D + r \cdot P]$ is of degree $r \geq g$. It follows from the Riemann-Roch theorem that $|D + r \cdot P|$ is a linear system of dimension ≥ 1. So there is an effective divisor D' of degree r with $D' \sim_{rat} D + r \cdot P$. Hence $\phi^P_{(r)}(D') = [D]$, which shows that $\phi^P_{(r)}$ is surjective.

Remark 2.4 The classes of W_r in $\mathrm{CH}_{\mathbb{Q}}(J(C))$ are related by the Pontryagin product by $[W_1]^{\star r} = [W_r] \cdot r!$ for $1 \leq r \leq g$. Indeed, in view of the following commutative diagram

$$\begin{array}{ccc} C \times C^{(r)} & \xrightarrow{\phi^P \times \phi^P_{(r-1)}} & J(C) \times J(C) \\ \downarrow {\scriptstyle (r:1)} & & \downarrow {\scriptstyle \mu} \\ C^{(r)} & \xrightarrow{\phi^P_{(r)}} & J(C) \end{array}$$

we have that $[W_1] \star [W_{r-1}] = r \cdot [W_r]$. Induction on r yields the result.

The case of $r = g - 1$ is of special interest: the subvariety W_{g-1} is a divisor and it has some remarkable properties. First of all, it gives a principal polarization on $J(C)$. Second, there is a translation of W_{g-1} that is a symmetric divisor.

Theorem 2.5 Let $\kappa \in J(C)$ be the image of the canonical divisor K under the map ϕ^P_{2g-2}. Let W^-_{g-1} be the pull-back of W_{g-1} by $[-1]$. Then

$$W^-_{g-1} = t^*_\kappa W_{g-1},$$

where $t_\kappa : J(C) \to J(C)$ is translating over κ.

Proof. See e.g. p.338 of [?] or Thm. A.8.2.1 of [?]. □

The map $[2] : J(C) \to J(C)$ is an isogeny of degree 2^{2g}. In particular, we can find an element $\lambda \in J(C)$ such that $2\lambda = \kappa$. Such an element is called a theta characteristic and it yields an identity

$$t^*_{-\lambda} W^-_{g-1} = t^*_\lambda W_{g-1}.$$

This divisor is symmetric. Indeed, we have $t^*_\lambda [-1]^* = [-1]^* t_{-\lambda}$, so

$$[-1]^* t^*_\lambda W_{g-1} = t^*_{-\lambda} W^-_{g-1} = t^*_\lambda W_{g-1}.$$

The fact that the variety W_{g-1} is a principal polarization is proven in e.g. Cor. 8.2.3 of [?].

We close this paragraph with an example that uses proposition 2.3, and that we need in section 3.

Example 2.6 (The difference map) Let F be a curve of genus 2. The map $\delta : F^2 \to J(F) : (P, Q) \mapsto [P - Q]$ is the difference map. It is independent of the choice of a point in F. Since F is hyperelliptic, we have the hyperelliptic involution $\iota : F \to F$ and this gives rise the map $(\mathrm{id}, \iota) : F \times F \to F \times F$. Composing with δ gives the map

$$F \times F \to J(F) : (P, Q) \mapsto [P - \iota(Q)].$$

But $[P - \iota(Q)] = [Q - \iota(P)]$, so the map is symmetric. Hence it factors through the symmetric product and we obtain a commutative diagram

$$\begin{array}{ccccc} F \times F & \xrightarrow{\delta} & J(F) & \xleftarrow{t_{-\kappa}} & J(F) \\ {\scriptstyle (id,\iota)} \uparrow & & \uparrow & \nearrow & \\ F \times F & \xrightarrow[\pi]{} & F^{(2)} & & \end{array}$$

The Ceresa cycles The Ceresa r-cycle is defined by $W_r - W_r^-$. The cycle $W_1 - W_1^-$ is sometimes called the basic cycle and some denote it with $C - C^-$. These cycles depend on a chosen point of P, but as we have seen, their geometric information does not depend on this choice. This is the reason for considering the classes of the Ceresa cycles in $\mathcal{A}_{\mathbb{Q}}(J(C))$, because these are independent of the choice of embedding of C in $J(C)$. The class of the r-cycle W_r in $\mathcal{A}_{\mathbb{Q}}(J(C))$ is denoted with w_r, or w^{g-r} if we want to stress codimension. There are a few occasions in which the class of a Ceresa cycle vanishes. We mention two of them.
First, for any nonsingular complete curve C of genus g, the image of the Ceresa $(g-1)$-cycle in $\mathcal{A}_{\mathbb{Q}}(J)$ is zero. Indeed, we may represent the class w_{g-1} by a symmetric divisor, showing that W_{g-1} and W_{g-1}^- have the same image in $\mathcal{A}_{\mathbb{Q}}(J(C))$.
Second, if C is hyperelliptic. Then there is a Weierstrass point $P \in C$, i.e. a point P such that the linear system $|2P|$ is a base-point free pencil. Furthermore, there is the hyperelliptic involution $\iota : C \to C$ which has the property that for every point Q, the divisor $Q + \iota(Q)$ is in $|2P|$. In other words,

$$\iota(Q) - P \sim_{rat} P - Q.$$

It follows that $W_1^- = (\phi^P \circ \iota)(C)$. But $\iota : C \to C$ is an isomorphism, so $W_1 = W_1^-$. So in this case, the image of the Ceresa 1-cycle in $\mathcal{A}_{\mathbb{Q}}(J(C))$ is trivial.

In section 3 we will see that the vanishing of the Ceresa cycles in $\mathcal{A}_{\mathbb{Q}}(J(C))$ is rather an exception than a rule. Before we can state and prove the various theorems there, we need some preliminary results.

2.2 Intermediate Jacobians

To a projective complex manifold X of dimension g we can associate the k-th intermediate Jacobian, for $k = 1, \ldots, g$. This is a complex torus that is constructed as follows. The manifold X has a polarization $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ which is the Chern class of an ample bundle on X. This form gives rise to the Hodge filtration[1]

$$H^k(X, \mathbb{C}) = F^0 H^k \supset F^1 H^k \supset \cdots \supset F^k H^k \supset F^{k+1} H^k = 0$$

of the cohomology group $H^k(X, \mathbb{C})$. In particular, for $2k-1$ we have

$$H^{2k-1}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^{2k-1}(X, \mathbb{C}) = F^k H^{2k-1}(X, \mathbb{C}) \oplus \overline{F^k H^{2k-1}(X, \mathbb{C})}$$

and this decomposition defines a complex structure on $H^{2k-1}(X, \mathbb{R})$. We find an isomorphism $H^{2k-1}(X, \mathbb{R}) \cong \overline{F^k H^{2k-1}(X, \mathbb{C})}$. For the manifolds of our interest, the cohomology groups $H^{2k-1}(X, \mathbb{Z})$ are without torsion. The k-th intermediate

[1] For the groups involved in the Hodge decomposition and filtration we use the (standard) notation of [**?**].

Jacobian is defined as the cokernel of the horizontal arrow below.

$$\begin{array}{ccc} H^{2k-1}(X,\mathbb{Z}) & \longrightarrow & \overline{F^k H^{2k-1}(X,\mathbb{C})} \\ & \searrow \quad \nearrow & \\ & H^{2k-1}(X,\mathbb{R}) & \end{array}$$

It is denoted with $J^k(X)$ and for $k=1$ and $k=g$, we get the Picard variety and the Albanese variety of X. For a curve C these varieties coincide, and moreover, $J^1(C)$ is isomorphic as a complex manifold, with $J(C)(\mathbb{C})$. This is the content of the famous theorems by Abel and Jacobi, which can be found in [**?**]. From now on, we will write $J(C)$ for these varieties.
Using Poincaré duality we find an alternative description:

$$J^{2k-1}(X) = \frac{(F^{g-k+1}H^{2g-2k+1}(X,\mathbb{C}))^*}{H_{2g-2k+1}(X,\mathbb{Z})}$$

and this description is suited to define the Abel-Jacobi map. This map is associates to certain algebraic cycles an element in $J^k(X)$. We use this image to study the cycle in question. More precise, consider the cycle class map

$$\mathrm{cl} : \mathscr{Z}_k(X) \to H^{2g-2k}(X,\mathbb{Z})$$

that associates to a variety its fundamental class. If the variety is singular, we use a desingularization to define a fundamental class. The kernel of this map is the group of algebraic cycles that are homologous to zero, denoted with $\mathscr{Z}^k(X)_{hom}$. The various subgroups of $\mathscr{Z}^k(X)$ we have defined so far are subgroups of each other:

$$\mathscr{Z}^k(X) \supset \mathscr{Z}^k(X)_{hom} \supset \mathscr{Z}^k(X)_{alg} \supset \mathscr{Z}^k(X)_{rat}.$$

The Abel-Jacobi map associates to a cycle $Z \in \mathscr{Z}^k(X)_{hom}$ the element $\int_\Gamma -$, which is integrating over a singular $(2g-2k+1)$-chain Γ whose boundary is Z. This gives a homomorphism

$$\Phi_X^k : \mathscr{Z}^k(X)_{hom} \to J^{2k-1}(X)$$

which is called the Abel-Jacobi map.

The intermediate Jacobian $J^k(X)$ contains an abelian variety[1] $J^k(X)_{alg}$, which is the largest subtorus whose tangent space is contained in $H^{k-1,k}(X)$, or Poincaré dual, in $H^{g-k+1,g-k}(X)^*$. The Abel-Jacobi map, when restricted to $\mathscr{Z}^k(X)_{alg}$, factors through this abelian variety, i.e. the diagram

$$\begin{array}{ccc} \mathscr{Z}^k(X)_{alg} & \xrightarrow{\Phi_X^k} & J^{2k-1}(X) \\ & \searrow \quad \nearrow & \\ & J^{2k-1}(X)_{alg} & \end{array}$$

commutes. This is a crucial observation, which is exploited in proofs of the next section. We give a motivation for this fact. For a $Z \in \mathscr{Z}^k(X)_{alg}$ we can find a curve S, a cycle $E \in \mathscr{Z}^k(X \times S)$ and two points $s,t \in S$ such that $E(s) - E(t) = Z$.

[1]We use the form $(\xi,\eta) \mapsto i^{2k-1}(-1)^{k-1}\int_{J(C)} \xi \wedge \overline{\eta} \wedge \omega \wedge \cdots \wedge \omega$. It is Hermitian form on $H^{2k-1}(J(C),\mathbb{C})$ that is positive definite on the summand $H^{k-1,k}(J(C))$. Its imaginary part takes integer values on the lattice $H^{2k-1}(J(C),\mathbb{Z})$, so we may invoke the Appell-Humbert theorem [**?**] to conclude that the subtorus is an abelian variety.

Since Z is homologous to zero, there is a $(2g-2k+1)$-cycle whose boundary is Z. We can choose this chain be the push-forward of a chain Γ lying on E. The image Z in $J^k(X)$ is independent of this choise so we find that $\Phi_X^k(Z)$ is integrating forms of type

$$(g-k+1, g-k) + (g-k+2, g-k-1) + \cdots$$

over a chain on a $(g-k+1)$-dimensional variety. This gives zero, except possibly for forms of type $(g-k+1, g-k)$.

In a similar fashion as above, we can define the primitive intermediate Jacobians. Let X be a g-dimensional projective complex manifold, with polarization ω as above. For $k = 1, \dots, g$, the primitive cohomology group $P_{\mathbb{Z}}^k(X)$ with values in $\mathbb{Z}$ is defined by the kernel of the Lefschetz operator

$$L : H^k(X, \mathbb{Z}) \to H^{2g-k}(X, \mathbb{Z}) : \xi \mapsto \xi \wedge \omega \wedge \cdots \wedge \omega.$$

Extending scalars to $\mathbb{C}$ yields $P_{\mathbb{C}}^k$. This group equals the kernel of the (complxified) Lefschets operator $L : H^k(X, \mathbb{C}) \to H^{2g-k}(X, \mathbb{C})$, and since L respects the bidegree, we see that $P_{\mathbb{C}}^k(X)$ admits a Hodge decomposition. It is compatible with the Hodge decomposition of $H^k(X, \mathbb{C})$ and it gives a filtration on $P_{\mathbb{C}}^k$. The k-th primitive intermediate Jacobian, where $k = 1, \dots, g$, is defined by the cokernel of the integration map

$$\lambda : H_{2g-2k+1}(X, \mathbb{Z}) \to (F^{g-k+1} P^{2g-2k+1}(X, \mathbb{C}))^*$$

and is denoted by $J^k(X)_{prim}$. This is a complex torus, and the largest subtorus whose tangent space is contained in $(P_{\mathbb{C}}^{g-k+1, g-k}(X))^*$ is called the algebraic primitive intermediate Jacobian, denoted by $J^k(X)_{prim,alg}$. This is again an abelian variety. The inclusion

$$P_{\mathbb{C}}^{2g-2k+1}(X) \subset H^{2g-2k+1}(X, \mathbb{C})$$

induces the surjective map $H^{2g-2k+1}(X, \mathbb{C})^* \to P_{\mathbb{C}}^{2g-2k+1}(X)^*$ and thus a projection $q : J^k(X) \to J^k(X)_{prim}$. The primitive Abel-Jacobi map is defined by sending an algebraic cycle $Z \in \mathscr{Z}^k(X)_{hom}$ to integration of primitive $(2g-2k+1)$-forms over a chain whose boundary is Z, modulo periods. This map is denoted by $\Psi_X^k : \mathscr{Z}^k(X)_{hom} \to J^k(X)_{prim}$ and it equals $q \circ \Phi_X^k$. Finally, the restriction of Ψ_X^k to $\mathscr{Z}^k(X)_{alg}$ factors through $J^k(X)_{prim,alg}$, i.e. the following diagram commutes.

$$\begin{array}{ccc} \mathscr{Z}^k(X)_{alg} & \xrightarrow{\Psi_X^k} & J^k(X)_{prim} \\ & \searrow \quad \nearrow & \\ & J^g(X)_{prim,alg} & \end{array}$$

We will use the (primitive) intermediate Jacobian to study the Ceresa 1-cycles. The advantage of this Jacobian is that the image

$$\Psi_{J(C)}^{g-1}(W_1 - W_1^-)$$

does not depend on the embedding $\phi^P : C \to J(C)$. A proof of this fact can be found in [?] or [?].

To show that the Ceresa 1-cycle is not algebraically equivalent to zero, it suffices to show that there is a closed (primitive) $(3,0)$-form on $J(C)$ such that the pairing of the (primitive) Abel-Jacobi image and this form is not equal to zero. This shows that the (primitive) Abel-Jacobi image lies outside the (primitive) algebraic intermediate Jacobian, so $C - C^- \notin \mathscr{Z}_1(J(C))_{alg}$.

REMARK 2.7 Consider a curve of the form $C = E + X$, where E is an elliptic curve, X a nonsingular curve of genus $g-1$ and where E and X have one point P in common. This is a curve of compact type and we describe its Jacobian variety (informally). For background of this construction we refer to the book [**?**] of Joe Harris and Ian Morrison.
For the holomorphic 1-forms we have $H^0(C, \Omega_C) = H^0(E, \Omega_E) \times H^0(X, \Omega_X)$. The first homology group doesn't change if we leave out P, i.e.

$$H_1(X, \mathbb{Z}) = H_1(X - \{P\}, \mathbb{Z}).$$

But $X - \{P\}$ is the disjoint union of $E - \{P\}$ and $X - \{P\}$. It follows that $H_1(C, \mathbb{Z}) = H_1(E, \mathbb{Z}) \times H_1(X, \mathbb{Z})$. We find that

$$J(C) = \frac{H^0(C, \Omega_C)^*}{H_1(C, \mathbb{Z})} = J(E) \times J(F).$$

2.3 Families

In the next subsection we will look at the variation of the Abel-Jacobi map in a family. To this end we have to fix some notations and collect some results concerning deformations of varieties, variation of Hodge structures and the sheaves $\mathscr{H}^k$.

Deformations. A deformation of a compact complex manifold is family[1] $\mathcal{X} \to B$ such that X occurs as the fiber over a specified point $0 \in B$. The other fibers $p^{-1}(b) = X_b$ are called deformations of $X = X_0$. Two deformations of X are called equivalent if they are isomorphic as families on open neighborhoods of the specified points. This shows that an equivalence class of a deformation of X can be represented by a family over an arbitrary small polydisc, such that X is the fiber over 0. To a deformation $\mathcal{X} \to B$ we can associate a short exact sequence

$$0 \to \mathscr{T}_X \to \mathscr{T}_{\mathcal{X}}|_X \to T_0B \otimes \mathscr{O}_X \to 0$$

of sheaves on X. The the first connecting homomorphism in the associated long exact sequence in cohomology is denoted with

$$\rho_0 : T_0B \to H^1(X, \mathscr{T}_X)$$

and is called the Kodaira-Spencer map. The elements in $H^1(X, \mathscr{T}_X)$ are called infinitesimal deformations. The Kodaira-Spencer map measures in a way, how far the family is off from being trivial. If the family trivial, then we can lift a vector field on the base without obstruction to one on $\mathcal{X}$. If the family is not trivial then there is obstruction in glueing the local liftings of the vector field. This obstruction lies in $H^1(X, \mathscr{T}_X)$. If the Kodaira-Spencer map is injective, the deformation is said to be effective. For a trivial family, the map ρ_0 is the zero map[2].
A deformation $\mathcal{X} \to B$ is complete at $0 \in B$ if every other deformation of X_0 is equivalent to the pull-back of $\mathcal{X} \to B$ by a morphism $B' \to B$. Such a pull-back is defined by the fibered square

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B. \end{array}$$

[1]By a family we mean a holomorphic proper submersion of connected complex manifolds.
[2]The converse of this statement is not true in general. A counter example is given by a deformation of Hopf manifolds, see e.g. 4.2(d) of [**?**]. Note that the family in this example is not flat.

A deformation is complete if it is complete at every point in B. Two important questions in deformation theory are (1) how can we decide whether a deformation is complete and (2) does such a complete deformation exist for a given manifold? These questions have been answered in the late fifties and early sixties by K. Kodaira, D.C. Spencer and M. Kuranishi.

THEOREM 2.8 (KODAIRA-SPENCER) If the Kodaira-Spencer map of a deformation $\mathcal{X} \to B$ is surjective for the point $0 \in B$ over which X occurs, then the deformation is complete at $0 \in B$.

PROOF. See the Theorem of Completeness in ([?]). □

The existence of a complete deformation was proved by Kuranishi.

THEOREM 2.9 (KURANISHI) For any compact complex manifold X there exist a complete analytic family $\mathcal{X} \to B$ such that X occurs as a fiber. This family is called the Kuranishi family of X.

PROOF. See the Theorem of Existence in ([?]). □

The inconvenience of the last theorem is that it is a theorem in the category of analytic spaces. This means that the base space need not be a complex manifold. The possible obstruction for being a manifold lies in the cohomology group $H^2(X, \mathscr{T}_X)$. For the cases in which we are interested there appears to be no obstruction. In the case of curves this cohomology group is zero. In the case of complex tori we can calculate what happens. Indeed, consider $X = \mathbb{C}^g/\Lambda_M$, where M is a matrix such that the imaginary part is positive definite and Λ_M is the lattice spanned by the columns of (I, M). The space of matrices with this property is a complex manifold and if we take B a small neighborhood of M in this space, we can construct a deformation of X over B. Indeed, the manifold that arises as the quotient of $\mathbb{C}^g \times B$ by the $\mathbb{Z}^{2g}$-action

$$\mathbb{Z}^{2g} \ni N : (z, M') \mapsto (z + (I, M')N, M')$$

is a deformation of X with fibers $\mathbb{C}^g/\Lambda_{M'}$. This is a complete and effective deformation of X. Note that for such a complete and effective deformation, the tangent space $T_M B$ can be identified with $H^1(X, \mathscr{T}_X)$ using the Kodaira-Spencer map.

REMARK 2.10 Another way of showing that a complex torus X of dimension g has a Kuranishi family $\mathcal{X} \to B$ with nonsingular base space B is the following. The cotangent bundle of X is trivial, which makes it a Calabi-Yau manifold. This is a compact Kähler manifold with trivial canonical bundle, i.e. $\mathscr{K}_X := \Omega^g_X \cong \mathscr{O}_X$. We can now invoke the Tian-Todorov lemma, to show that every infinitesimal deformation is integrable. For a brief discussion and references, see e.g. [?].

Not every complex torus is an abelian variety. The Riemann relations imply that $X = \mathbb{C}^g/\Lambda_M$ is an abelian variety if M is in addition symmetric. Indeed, to be an abelian variety, X needs an ample line bundle. We can construct one if M is symmetric with positive imaginary part, using the Appell-Humbert theorem (see e.g. [?]). Suppose that $X = \mathbb{C}^g/\Lambda_M$ is an abelian variety. Let B be as above, and let $B' \subset B$ be the closed subvariety consisting of the symmetric matrices. This gives a deformation of X such that all the fibers are abelian varieties. This deformation is complete as a deformation of abelian varieties. Moreover, B' can be viewed of an open neighborhood of the class of X is de moduli space of polarized abelian varieties. The image of the Kodaira-Spencer map consists of the symmetric endomorphisms

of $\mathbb{C}^g$. We formalize this as follows.
A polarized abelian variety is a pair (X, ω), where X is an abelian variety and where $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is the first Chern class of an ample line bundle on X. A deformation of a (X, ω) is a deformation $\mathcal{X} \to B$ of X with a polarization $\Omega \in H^2(\mathcal{X}, \mathbb{Z})$ that restricts to ω on X. We are interested in the image of the Kodaira-Spencer map of this deformation.
Such a polarization gives an isogeny $\varphi_\omega : X \to X^t$ of X with its dual abelian variety X^t. Its differential $d\varphi_\omega(o) : T_oX \to T_oX^t$ at the identity $o \in X$ is an isomorphism and using the identifications $T_oX^t = H^1(X, \mathscr{O}_X)$ and $T_oX = H^0(X, \mathscr{T}_X)$, we get a commutative diagram

$$\begin{array}{ccc} T_0X & = & H^0(X, \mathscr{T}_X) \\ {\scriptstyle d\varphi_\omega(o)}\downarrow & & \downarrow{\scriptstyle \widehat{\omega}} \\ T_oX^t & = & H^1(X, \mathscr{O}_X), \end{array}$$

where the right vertical map is cupping with ω. The tangent sheaf $\mathscr{T}_X$ is free, i.e. we have $\mathscr{T}_X = T_oX \otimes \mathscr{O}_X$. It follows that $H^1(X, \mathscr{T}_X) = H^1(X, \mathscr{O}_X) \otimes T_oX$, which in turn is isomorphic to $H^1(X, \mathscr{O}_X) \otimes H^1(X, \mathscr{O}_X)$ via the map $\mathrm{id} \otimes \widehat{\omega}$. This must hold for every fiber in the family, and this imposes a symmetric condition on the image of ρ_0. This is reflected in the fact that ρ_0 maps onto the symmetric part of $H^1(X, \mathscr{T}_X)$. The symmetric part can be identified with the kernel of the natural projection

$$H^1(X, \mathscr{O}_X) \otimes H^1(X, \mathscr{O}_X) \to \bigwedge^2 H^1(X, \mathscr{O}_X).$$

Using the identification $H^2(X, \mathscr{O}_X) = H^1(X, \mathscr{O}_X) \bigwedge H^1(X, \mathscr{O}_X)$, we get the commutative square

$$\begin{array}{ccc} H^1(X, \mathscr{T}_X) & \xrightarrow{\ \widehat{\omega}\ } & H^2(X, \mathscr{O}_X) \\ {\scriptstyle id\otimes\widehat{\omega}}\downarrow & & \| \\ H^1(X, \mathscr{O}_X) \otimes H^1(X, \mathscr{O}_X) & \longrightarrow & \bigwedge^2 H^1(X, \mathscr{O}_X). \end{array}$$

The kernel of the upper horizontal map (cupping with ω) is denoted with

$$H^1(X, \mathscr{T}_X)_\omega.$$

It consists of the infinitesimal deformations that might occur in the image of the Kodaira-Spencer map of a complete effective deformation of the polarized abelian variety (X, ω). The fact that they actually do occur, follows from the fact that there is a complete and effective deformations of the pair (X, ω). It follows that the base space is an open neighborhood of the class $[X]$ of X in $\mathfrak{A}_g$, the moduli space of g-dimensional polarized abelian varieties over $\mathbb{C}$. The tangent space of $\mathfrak{A}_g$ at $[X]$ is identified with $H^1(X, \mathscr{T}_X)$. Rigorous proofs of the above statements about deformations of abelian varieties can be found in [**?**].

The period map. The complex structure on a manifold is closely related to the Hodge structure. The way in which this structure varies in a family is described with the period map. For a deformation $\mathcal{X} \to B$ of a compact complex manifold X with B small enough, we see that the cohomology groups $H^k(X_b, \mathbb{C})$ are all isomorphic. With these isomorphisms we can map the Hodge filtrations of the cohomology of a fiber, to the flag manifold of the filtrations of the cohomology $H^k(X, \mathbb{C})$ of the central fiber X. This map is the holomorphic period map

$$\mathcal{P}^k : B \to \mathrm{Flag}(b^{k,k}, b^{k,k-1}, \ldots, b^{k,0}; H^k(X, \mathbb{C})).$$

This flag manifold appears as a submanifold of the product of the Grassmann varieties, i.e.

$$\mathrm{Flag}(b^{k,k}, b^{k,k-1}, \dots, b^{k,0}; H^k(X,\mathbb{C})) \subset \prod_{p=0}^{k} \mathrm{Grass}(b^{k,p}, H^k(X,\mathbb{C})).$$

The period map factors through this product, and projecting onto its factors gives the maps

$$\mathcal{P}^{k,p} : B \to \mathrm{Grass}(b^{k,p}, H^k(X,\mathbb{C})), \quad k = 0, \dots, k.$$

These are also holomorphic and the differentials at $b \in B$ are linear maps

$$d\mathcal{P}^{k,p}(b) : T_bB \to \mathrm{Hom}_{\mathbb{C}}(F^pH^k(X_b,\mathbb{C}), H^k(X_b,\mathbb{C})/F^pH^k(X_b,\mathbb{C})).$$

Investigation of these period maps shows that the the differential $d\mathcal{P}^{k,p}(b)$ factors, i.e. the following diagram commutes.

$$\begin{array}{ccc} & & \mathrm{Hom}_{\mathbb{C}}(F_b^p/F_b^{p+1}, F_b^{p-1}/F_b^{p-2}) \\ & \nearrow & \downarrow \\ T_bB & \xrightarrow[d\mathcal{P}^{k,p}(b)]{} & \mathrm{Hom}_{\mathbb{C}}(F_b^p, H^k/F_b^p) \end{array} \tag{2.1}$$

In view of the identifications $H^{p,q}(X_b) \cong H^q(X, \Omega^p_{X_b})$, we arrive at the commutative diagram

$$\begin{array}{ccc} T_bB & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(F^pH^k(X_b,\mathbb{C}), H^k(X_b,\mathbb{C})/F^pH^k(X_b,\mathbb{C})) \\ \downarrow{\scriptstyle \rho_b} & & \uparrow \\ H^1(X_b, \mathscr{T}_{X_b}) & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(H^{k-p}(X_b, \Omega^p_{X_b}), (H^{k-p+1}(X_b, \Omega^{p-1}_{X_b}))). \end{array}$$

The upper horizontal arrow is $d\mathcal{P}^{k,p}(b)$, the lower sends an element θ to "cupping with θ".

In the case of the Kuranishi family $\mathcal{X} \to B$ of a complex torus of dimension g, we may consider the period map $\mathcal{P}^{g,g}$. Since $F^gH^g(X,\mathbb{C}) = H^0(X, \Omega^g_X)$ is of dimension one, this is a map

$$\mathcal{P}^{g,g} : B \to \mathbb{P}(H^g(X,\mathbb{C})).$$

This map is an immersion and we say that "the periods give local moduli". The proof can be found in 10.3.2 of [?]. This results in an isomorphism

$$H^1(X, \mathscr{T}_X) \to \mathrm{Hom}_{\mathbb{C}}(H^0(X, \Omega^g_X), H^1(X, \Omega^{g-1}_X)),$$

or equivalently, an isomorphism

$$\mu : H^1(X, \mathscr{T}_X) \otimes H^0(X, \Omega^g_X) \to H^1(X, \Omega^{g-1}_X),$$

where μ is induced by the cup product.
Our main interest however, lies at abelian varieties and their deformations. The infinitesimal deformations of a polarized abelian variety (X, ω) of dimension g are paramatrized by $H^1(X, \mathscr{T}_X)_\omega$. To see how μ acts on $H^1(X, \mathscr{T}_X)_\omega \otimes H^0(X, \Omega^g_X)$ we first consider the commutative diagram

$$\begin{array}{ccc} H^1(X, \mathscr{T}_X) \otimes H^0(X, \Omega^g_X) & \xrightarrow{\widehat{\omega} \otimes id} & H^2(X, \mathscr{O}_X) \otimes H^0(X, \Omega^g_X) \\ {\scriptstyle \mu}\downarrow & & \downarrow \\ H^1(X, \Omega^{g-1}_X) & \xrightarrow{L} & H^2(X, \Omega^g_X). \end{array}$$

The map $\widehat{\omega}$ is cupping with ω and L is the Lefschetz operator. The right vertical map is an isomorphism. It follows that μ', the restriction of μ to $H^1(X, \mathscr{T}_X)_\omega \otimes H^0(X, \Omega_X^g)$, takes values in the primitive part $H^1(X, \Omega_X^{g-1})_0$. Moreover, we see that

$$\mu' : H^1(X, \mathscr{T}_X)_\omega \otimes H^0(X, \Omega_X^g) \to H^1(X, \Omega_X^{g-1})_0$$

is surjective. This is an important observation.

All the primitive forms of type $(g-1, 1)$ can be obtained by taking a $(g, 0)$-form and cup it with some infinitesimal deformation of (X, ω).

The sheaves $\mathscr{H}^k$. Let $\pi : \mathcal{X} \to B$ be a deformation of the polarized abelian variety (X, ω). Instead of looking at the period maps, we can also study the variation of the Hodge structures via the sheaves

$$\mathscr{H}^k := R^k \pi_* \mathbb{C} \otimes \mathscr{O}_B.$$

The sheaf $\mathscr{H}^k$ is locally free, so we may view it as the sheaf of sections of a vector bundle. The fiber of this bundle over $b \in B$ is isomorphic to the cohomology group $H^k(X_b, \mathbb{C})$. Furthermore, $\mathscr{H}^k$ has a filtration that corresponds to the Hodge filtration on the fibers, i.e. there are locally free subsheaves

$$0 = F^{k+1}\mathscr{H}^k \subset F^k \mathscr{H}^k \subset \ldots \subset F^1 \mathscr{H}^k \subset F^0 \mathscr{H}^k = \mathscr{H}^k$$

and the inclusion $F^p \mathscr{H}^k \subset \mathscr{H}^k$ corresponds to $F^p H^k(X_b \mathbb{C}) \subset H^k(X_b, \mathbb{C})$ on the fibers.
The sheaf $\mathscr{H}^k$ carries a flat connection

$$\nabla : \mathscr{H}^k \to \mathscr{H}^k \otimes \Omega_B$$

called the Gauss-Manin connection. It satisfies Griffiths transversality

$$\nabla : F^p \mathscr{H}^k \to F^{p-1} \mathscr{H}^k \otimes \Omega_B.$$

An important aspect of this construction is the following. Given a deformation $\pi : \mathcal{X} \to B$ of X and a class ξ_0 in $F^p H^k(X, \mathbb{C})$, we can view this class as a varying class, i.e. the value of a local section ξ of $F^p \mathscr{H}^k$. Its values are denoted with $\xi_b \in F^p H^k(X_b, \mathbb{C})$. Using the isomorphisms $H^k(X_b, \mathbb{C}) \cong H^k(X, \mathbb{C})$, this gives a map $x : B \to H^k(X, \mathbb{C})$ that depends holomorphically on b. We want to know its derivative in the direction of v, a vector field on B. This can be calculated with the formula:

$$v(x)(b) = \nabla_v(\xi)|_b = d\mathcal{P}^{k,p}(b)(v(b))(\xi_b),$$

where $\nabla_v(\xi)$ is the evaluation of $\nabla \xi$ on v. The right hand side is an element of $F^{p-1} H^k(X_b, \mathbb{C})$ which gives a class in the quotient $H^{p-1, k-p+1}(X_b, \mathbb{C})$. With the identifications of $H^{p,q}(X_b) \cong H^q(X_b, \Omega_{X_b}^p)$, this corresponds to

$$\rho_b(v(b)) \cdot \xi_b \in H^{k-p+1}(X_b, \Omega_{X_b}^{p-1}).$$

The Gauss-Manin connection is also compatible with Poincaré duality. More precisely, if ν is a section of $\mathscr{H}^{k*}$ and η is a section of $\mathscr{H}^k$, then ν corresponds to a section ν' of $\mathscr{H}^{g-k}$ and the pairing $\nu(\eta) = \langle \nu', \eta \rangle$ is a section of $\mathscr{O}_B$. For a vector field v on B we can calculate the directional derivative

$$v(\nu(\eta)) = \langle \nabla_v \nu', \eta \rangle + \langle \nu', \nabla_v \eta \rangle. \tag{2.2}$$

2.4 Variation of the Abel-Jacobi map

For a polarized complex manifold X we have defined the various intermediate Jacobians, and they behave nicely in a family. More precise, for a deformation $\pi : \mathcal{X} \to B$ of X we have the sheaf $\mathscr{H}^{2k-1}$ with subsheaf $F^k\mathscr{H}^{2k-1}$. The exact sequence

$$0 \to F^k\mathscr{H}^{2k-1} \to \mathscr{H}^{2k-1} \to \mathscr{L}ie(\mathscr{J}^k) \to 0$$

defines the sheaf $\mathscr{L}ie(\mathscr{J}^k)$ on B. If we denote the locally constant sheaf $R^k\pi_*\mathbb{Z}$ with $\mathscr{H}^{2k-1}_{\mathbb{Z}}$, we get the defining sequence

$$0 \to \mathscr{H}^{2k-1}_{\mathbb{Z}} \to \mathscr{L}ie(\mathscr{J}^k) \to \mathscr{J}^k \to 0,$$

which we stipulate to be exact. The sheaf $\mathscr{J}^k$ may be thought of as the sheaf of sections of the family of k-th intermediate Jacobians of the fibers over B.
Let's denote this family with $\{J^k(X_b)\}_b$ for a moment. The manifold $\mathcal{X}$ is also polarized and hence projective. Let $Z \in \mathscr{Z}^k(X)$ be a cycle that is flat over B. If we intersect with the fibers X_b, we see that Z may be considered as a family of algebraic cycles $\{Z_b\}_b$ where

$$Z_b = X_b \cdot Z \in \mathscr{Z}^k(X_b).$$

If in addition, each Z_b is homologous to zero, we can associate to every point $b \in B$ the Abel-Jacobi image of Z_b. In this way we get a map

$$B \to \{J^k(X_b)\}_b : b \mapsto \Phi^k_{X_b}(Z_b).$$

This map turns out to be holomorphic, which was first proven by Griffiths. So it is a section of the sheaf $\mathscr{J}^k$, and as such, it is denoted with ν_Z.
Suppose that B is a projective variety. Then the section ν_Z possesses yet another property: it is normal. This means that it is a section of the subsheaf of normal functions, which is defined as follows. The Gauss-Manin connection induces a morphism

$$\nabla : \mathscr{J}^k \to \mathscr{H}^{2k-1}/F^{k-1}\mathscr{H}^{2k-1} \otimes \Omega_B$$

and the sheaf of normal functions is the kernel of this morphism.
One consequence of ν_Z being normal is the following. Locally, we can lift ν to a section $\widetilde{\nu}$ of $\mathscr{H}^{2k-1}$, and application of ∇ yields

$$\nabla\widetilde{\nu} \in F^{k-1}\mathscr{H}^{2k-1} \otimes \Omega^B. \tag{2.3}$$

In fact, this holds for every lift of ν_Z. Condition (2.3) is equivalent to

$$\langle \nabla_v\widetilde{\nu}, \eta\rangle = 0, \quad \text{for all } \eta \in F^{n-k+2}\mathscr{H}^{2n-2k+1}$$

and all vector fields v on B. In view of (2.2), this shows that any local lift $\widetilde{\nu}$ of the normal function ν satisfies

$$v(\langle\widetilde{\nu}, \eta\rangle) = \widetilde{\nu}(\nabla_v\eta) \tag{2.4}$$

for every vector field v on B and every section η of $F^{n-k+2}\mathscr{H}^{2n-2k+1}$. Another consequence of ν_Z being normal, is that we can consider its infinitesimal invariant. We discuss this in subsection 3.3.

Finally, we note the existence of the families

$$\{J^k(X_b)_{alg}\}_b, \quad \{J^k(X_b)_{prim}\}_b \quad \text{and} \quad \{J^k(X_b)_{prim,alg}\}_b$$

whose sheaves of sections we denote with respectively

$$\mathscr{J}^k_{alg}, \quad \mathscr{J}^k_{prim} \quad \text{and} \quad \mathscr{J}^k_{prim,alg}.$$

The projection $J^k(X) \to J^k(X)_{prim}$ and the inclusions of the abelian subvarieties

$$J^k(X)_{alg} \subset J^k(X) \quad \text{and} \quad J^k(X)_{prim,alg} \subset J^k(X)_{prim}$$

are reflected by the sheaf morphism $q : \mathscr{J}^k \to \mathscr{J}^k_{prim}$ and the inclusions

$$\mathscr{J}^k_{alg} \subset \mathscr{J}^k \quad \text{and} \quad \mathscr{J}^k_{prim,alg} \subset \mathscr{J}^k_{prim}.$$

The section ν_Z, or rather the cycle Z, gives rise to a section $\nu_{0,Z}$ of $\mathscr{J}^k_{prim}$. It corresponds to the holomorphic map

$$B \to \{J^k(X_b)_{prim}\}_b : b \mapsto \Psi^k_{X_b}(Z_b).$$

If the cycles Z_b are in $\mathscr{Z}^k(X_b)_{alg}$, then ν_Z and $\nu_{0,Z}$ can be seen as sections of the sheaves $\mathscr{J}^k_{alg}$ and $\mathscr{J}^k_{prim,alg}$ respectively.

Remark 2.11 We close this section with a different kind of variation of the Abel-Jacobi map. Let X be a complex projective variety of dimension g and let S be a nonsingular connected curve. Consider an algebraic cycle $Z \in \mathscr{Z}^k(X \times S)$ all of whose components map onto S. Fix a base point $s_0 \in S$ and define

$$\nu_Z : S \to J^{2k-1}(X) : s \mapsto \Phi^k_X(Z(s) - Z(s_0)).$$

Griffiths has shown that this map is holomorphic. See e.g. Thm. 12.4 of [**?**].

3 Theorem of Ceresa

In the previous section we have defined the Ceresa cycles $W_r - W_r^-$ on a Jacobian variety of a curve of genus g. We have seen the vanishing of the class of the Ceresa r-cycle in two cases. First for the Ceresa $(g-1)$-cycle, since it has a symmetric translate. Second for the cycle $C - C^-$ in case C is hyperelliptic. In the next section we will show that all the Ceresa r-cycles vanish in the algebraic equivalent ring if C is hyperelliptic. But as we mentioned before, these cases are exceptions.

Theorem 3.1 (Ceresa,1983) On a generic Jacobian variety $J(C)$ of dimension $g \geq 3$, the algebraic cycles

$$W_r - W_r^-, \quad 1 \leq r \leq g-2$$

are not algebraically equivalent to zero.

The proof that was given by Ceresa in [?] takes three steps. In *step 1* he shows that if the theorem is not true for $g = 3, r = 1$, then the normal function ν_0 with respect to the Ceresa cycle and values in the 2^{nd} p rimitive intermediate Jacobian, is the zero function. In *step 2* he produces a counterexample for this statement. In *step 3* he generalizes to the case $g \geq 3$ and $1 \leq r \leq g-2$ using an induction argument.

In the same year that Ceresa's result was published, Bruno Harris produced an article [?] in which he analyses the normal function we described above. He finds a way to express the value of this normal function with iterated integrals, using the concept of a "harmonic volume". Furthermore, he shows that the differential of ν_0 is not equal to zero. This is another counterexample of *step 1* of Ceresa's proof. In his article [?] he uses his formulae to calculate explicitly the value of ν_0 at the Fermat curve of genus 3, paired with a non-trivial $(3,0)$-form. This turns out to be nonzero, which shows that for this curve the Ceresa 1-cycle is not algebraically equivalent to zero. This is another counter example to the conclusion of *step 1*.
The techniques of Harris cannot be used prove to Ceresa's theorem for $g \geq 3, r = 1$. The normal function ν_0 need not be zero if $C - C^- \sim_{alg} 0$, for $g \geq 4$. So the non-vanishing of the differential of ν_0 is not a counterexample for curves of higher genera.

This problem is solved in the paper [?] by Collino and Pirola. They note that the normal function ν_0 can be viewed as a section of the family of algebraic primitive intermediate Jacobians if Ceresa's theorem is not true for $g \geq 3, r = 1$. In this case, the infinitesimal invariant (which we will discuss later) $\delta(\nu_0)$ of ν_0 can be seen as a trivial functional on some vector space K'. However, they show that for every non-hyperelliptic curve, the functional $\delta(\nu_0)$ acts nontrivially on K'. This proves Ceresa's theorem for $g \geq 3, r = 1$.

3.1 Ceresa's proof

Before we start the proof, we take a closer to the statement of the theorem. This way of looking at the theorem was pointed out to me by Frans Oort and it clarifies what happens if the theorem is not true in the case of $g = 3, r = 1$.

An observation involving a Chow variety. Let $\mathfrak{M}_3$ be the moduli space of nonsingular curves over $\mathbb{C}$ of genus 3. We want to apply techniques of section 2, so we need a universal family of curves. This doesn't exist over $\mathfrak{M}_3$, so we consider the fine moduli space $\mathfrak{M}_{3,N}$ of curves of genus 3 with some level N-structure[1]. This

[1] $N \geq 3$ suffices

gives a finite covering

$$\mathfrak{M}_{3,N} \to \mathfrak{M}_3$$

and over $\mathfrak{M}_{3,N}$ there is a universal family $\pi : \mathcal{C} \to \mathfrak{M}_{3,N}$. For this family, there is a family of Jacobian varieties $\mathcal{J} \to \mathfrak{M}_{3,N}$, and this is a projective family. By this we mean that both $\mathcal{J}$ and $\mathfrak{M}_{3,N}$ are projective.
The fibers of these families over a point $[C]$ give the curve C and its Jacobian variety $J(C)$. Suppose that the Ceresa 1-cycle is algebraically equivalent to zero for this curve. Then there is a curve S_C, a positive 2-cycle $E_C \in \mathscr{Z}_2(J(C))$, a positive 1-cycle D_C on $J(C)$ and two points s_C, t_C in S such that

$$E_C(s_C) = W_1(C) + D_C \quad \text{and} \quad E_C(t_C) = W_1^-(C) + D_C. \tag{3.1}$$

The cycle E_C can be seen as a 2-cycle in some fixed projective space, the curve S can be embedded as a curve in $\mathbb{P}^3$, and the cycle D_C is a 1-cycle on the fiber of a family of projective varieties. These objects can be paramatrized by a Chow variety $\mathfrak{Chow}$, and it obviously consists of many components. We can now look for a point (E, S, D_C) in $\mathfrak{Chow}$ if there are s, t in S such that (3.1) holds. This is a locally closed condition on the points of $\mathfrak{Chow}$, i.e. the points for which this holds form a locally closed set V of $\mathfrak{Chow}$. The image V' of V of the projection morphism $\mathfrak{Chow} \to \mathfrak{M}_3$ is constructable. This follows from the observation that a locally closed set is constructible and a theorem by Chevalley, which says that the image of a constructible set is constructible. For proofs and references, see e.g. [**?**], Exercise II.3.19. We distinguish two cases for the dimension of V'.

Case 1 We have $\dim V' = \dim \mathfrak{M}_3$. In this case, there is a Zariski open set $\mathfrak{U} \subset \mathfrak{M}_3$ contained in V' and families

$$\{E_C\}_{[C]}, \quad \{S_C\}_{[C]} \quad \text{and} \quad \{D_C\}_{[C]}$$

over $\mathfrak{U}$ of positive 2-cycles E_C on $J(C) \times S_C$, curves S_C and 1-cycles D_C on $J(C)$ that vary algebraically with $[C]$ and for which we can find points s_C, t_C such that (3.1) holds.

Case 2 We have $\dim V' < \dim \mathfrak{M}_3$, in which case there are countably many subvarieties of $\mathfrak{M}_3$ such that for points outside their union, the Ceresa 1-cycle is not algebraically equivalent to zero.

The theorem of Ceresa says that if $g = 3, r = 1$, we are in Case 2. If we assume that Ceresa's theorem is not true then Case 1 applies.

Proof of Ceresa's theorem. As we mentioned before, we proceed in three steps. The adjective 'open' means analytically open.

Step 1. Suppose that the theorem is not true for $g = 3, r = 1$, i.e. we are in Case 1. The pre-image of the Zariski open set $\mathfrak{U} \subset \mathfrak{M}_3$ in $\mathfrak{M}_{3,N}$ is denoted with $\mathfrak{U}'$. The family of curves $\{S_C\}_{[C]}$ is denoted with $\mathcal{S} \to \mathfrak{U}'$. The family $\mathcal{J} \to \mathfrak{M}_{3,N}$ gives rise the families of (primitive) 2^{nd} intermediate Jacobians of the fibers of $\mathcal{J}$. The sheaves of sections are denoted with $\mathscr{J}^2_{prim}$ and $\mathscr{J}^2$. The family $\mathcal{C} \to \mathfrak{M}_{3,N}$ has local sections and these are used to embed the curves in their Jacobians. More precise, a local section $P : U \to \mathcal{C}$ on $U \subset \mathfrak{M}_{3,N}$ gives a map $\phi^P : \mathcal{C}_U \to \mathcal{J}_U$. The image is a flat cycle $\{W_1(C)\}_{[C]}$ in $\mathcal{J}_U$ over U. Using the inverse on each fiber, we can construct the flat cycle

$$\{W_1(C) - W_1^-(C)\}_{[C]}$$

over U. This gives local sections ν and ν_0 of $\mathscr{J}^2$ and $\mathscr{J}^2_{prim}$. Since the Abel-Jacobi image $\Psi^2_{J(C)}(W_1(C) - W_1^-(C))$ is independent of the choice of embedding

$C \to J(C)$, we see that the local sections ν_0 glue to a global section over $\mathfrak{M}_{3,N}$. This section is holomorphic and moreover, it is the zero section. We present two proofs for this fact. The first one was used by Ceresa, the second follows from observations in the previous section.

(1) Let U be an open set in $\mathfrak{U}'$, and consider the sections ν and ν_0. Note that ν is a lift of ν_0 if U is small enough. It is sufficient to show that ν_0 is zero on U since it will then be zero on $\mathfrak{U}$. But then it is automatically zero on the whole of $\mathfrak{M}_{3,N}$ since $\mathfrak{U}$ is a dense subset.

Assume $[C] \in U$. The value of $\nu_0([C])$ is $\Psi^2_{J(C)}(W_1(C) - W_1^-(C))$, and in view of (3.1) and remark 2.11, we have

$$\Psi^2_{J(C)}(W_1(C) - W_1^-(C)) = \psi_{s_C}(t_C).$$

The map $\psi_{s_C} : S_C \to J^2(J(C))_{prim}$ is holomorphic and moreover, it lifts to the universal cover of $J^2(J(C))_{prim}$. This shows that ϕ_{s_C} is the zero map. Indeed, the lift is holomorphic and S_C is a compact Riemann surface so we can apply the maximum principle. To see that there is a lift, we show that the induced map in homology is trivial. To this end, consider a cycle class $\Gamma \in H_1(S_C, \mathbb{Z})$. Let $\widetilde{\Gamma} \in H_3(E_C, \mathbb{Z})$ be the cycle class of the inverse image of Γ. This in turn can be pushed forward to a cycle class $\overline{\Gamma} \in H_3(J(C), \mathbb{Z})$. Using Poincaré duality

$$PD : H_3(J(C), \mathbb{Z}) \to H^3(J(C), \mathbb{Z}),$$

this gives an element $\gamma_C \in H^3(J(C), \mathbb{Z})$. We may view this as the value of a local section γ of the sheaf $\mathscr{H}^3_{\mathbb{Z}}$. In particular, for every vector field u on U, we have $\nabla_u(\gamma) = 0$. Let $\xi_C \in F^3H^3(J(C), \mathbb{C}) = H^{3,0}(J(C))$ be the value at $[C]$ of a local section ξ of $F^3\mathscr{H}^3$. By consideration of type, or by noting that $\nu([C]) \in J^2(J(C))_{alg}$, it follows that the pairing $\langle \gamma, \xi \rangle = 0$. This pairing is a local section of $\mathscr{O}_{\mathfrak{M}_{g,N}}$, and application of the derivation u yields

$$0 = u(\langle \gamma, \xi \rangle) = \langle \nabla_u(\gamma), \xi \rangle + \langle \gamma, \nabla_u(\xi) \rangle = \langle \gamma, \nabla_u(\xi) \rangle.$$

Evaluated at $[C]$, the section $\nabla_u(\xi)$ is an element of $F^2H^3(J(C), \mathbb{C})$. It can be written as a sum $\xi'_C + \xi''_C$, where the first term is in $H^{3,0}(J(C))$ and the second is in $H^{2,1}(J(C))$. Since

$$\langle \gamma_C, \xi'_C \rangle = \int_{\overline{\Gamma}} \xi'_C = 0,$$

we arrive at the conclusion that $\int_{\overline{\Gamma}} \xi''_C = 0$.

But $\mathcal{J}_U \to U$ is a family of principally polarized abelian varieties over $\mathbb{C}$ of dimension 3. So it arises as the pull-back of the family over a subset of the moduli space $\mathfrak{A}_3$ of principally polarized abelian varieties over $\mathbb{C}$, by the induced map

$$U \to \mathfrak{A}_3.$$

This map is an injection by the Torelli theorem[1]. The dimensions of U and $\mathfrak{A}_3$ spaces are equal. Indeed, for $g \geq 2$ we have

$$\dim \mathfrak{M}_g = 3g - 3 \quad \text{and} \quad \dim \mathfrak{A}_g = \binom{g+1}{2}.$$

So we may consider $\mathcal{J}_U \to U$ as the Kuranishi family of $J(C)$. We now invoke an observation of the previous section:

All the primitive forms of type $(2,1)$ can be obtained by taking a $(3,0)$-form and

[1] This theorem states that the induced map $\mathfrak{M}_g \to \mathfrak{A}_g$ is injective.

cup it with some infinitesimal deformation of $J(C)$.

So the class of ξ''_C corresponds to a primitive form

$$\rho_C(u([C])) \cdot \xi_C,$$

and by varying ξ and u, we can obtain every primitive $(2,1)$-form in this way. This shows that

$$\int_{\overline{\Gamma}} \eta = 0$$

for every class $\eta \in F^2 P^3_{\mathbb{C}}(J(C))$. So the induced map $\psi_{s_C *}$ in homology is zero, showing that ψ_{s_C} lifts to $(F^2 P^3_{\mathbb{C}}(J))^*$. This establishes the claim that ψ_{s_C} is the zero function. This holds for every point $[C] \in U$, so ν_0 is the zero section over U.
(2) Another way of showing that ν_0 is the zero section, is to exploit the fact that ν is normal. Let $U \subset \mathfrak{M}_{3,N}$ be an open set lying in $\mathfrak{U}'$ and let $\widetilde{\nu}$ be a (local) lift of ν to $\mathscr{H}^3$. Let ξ be a section of $F^3\mathscr{H}^3$ over U. Since ν is normal, it follows from (2.4) that we have

$$u(\langle \widetilde{\nu}, \xi \rangle) = \widetilde{\nu}(\nabla_v \xi)$$

for every vector field u on U. By considerations of type we conclude that $\widetilde{\nu}(\nabla_v \xi) = 0$. Reasoning in the same fashion as above, we conclude that ν_0 is the zero section on U.
Note that ν_0 being the zero section is exclusive for the case $g = 3$, since we need all the deformations of $J(C)$ to arrive at this conclusion. For higher genera we have $\dim \mathfrak{M}_g < \dim \mathfrak{A}_g$, in which cases we cannot use this reasoning.

Step 2. Here we present the counterexample for ν_0 being the zero section on $\mathfrak{M}_{3,N}$. First we consider the local sections[1] ν_0 on open pieces of the moduli space $\mathfrak{M}_3$. These are also zero sections, since they agree with restrictions of the zero section ν_0 we defined on $\mathfrak{M}_{3,N}$. In fact, we can define these local sections on $\overline{\mathfrak{M}}_3$, the moduli space consisting of classes of curves whose Jacobian is an abelian variety. These local sections must be zero too.
Consider the connected curve $C_0 = E + F$, where E is an elliptic curve and F a nonsingular complete connected curve of genus 2 (see remark 2.7). We assume that the intersection $E \cap F = \{P_0\}$ is not a Weierstrass point of F. The Jacobian variety $J(C_0)$ degenerates:

$$J(C_0) = J(E) \times J(F).$$

The polarizing classes of $J(E)$ and $J(F)$ are denoted with ω_E and ω_F respectively. The polarizing class of $J(C_0)$ is then $\omega_E + \omega_F$. Any class $\xi \in H^{1,0}(J(F))$ gives rise to a class $\eta = \xi \wedge (\omega_E - \omega_F)$ and this is a primitive form. The Ceresa 1-cycle can be written as

$$(E \times \{0\} + \{0\} \times F) - (E \times \{0\} + \{0\} \times F^-) = (\{0\} \times F) - (\{0\} \times F^-),$$

where we use the point P_0 to embed the curves. The curve F^- is the same as the image of the embedding F in $J(F)$ using the point $\iota(q)$. Indeed, we have

$$F^- = -\phi^{P_0}(F) = (\phi^{\iota(P_0)} \circ \iota)(F) = \phi^{\iota(P_0)}(F).$$

Consider the map $F \times F \to J(F) \times F : (P, Q) \mapsto ([P - Q], Q)$. Its image is a cycle in $J(F) \times F$ such that intersection with a horizontal slice yields $\phi^Q(F)$. In particular, if Γ is a path on F from P_0 to $\iota(P_0)$, then the image of $F \times \Gamma$ in $J(F) \times F$, projected

[1]There is no global section, since there is no family of Jacobian varieties over $\mathfrak{M}_g$. This is due to the absence of a universal curve over this space.

onto $J(F)$ is a singular chain Γ' with boundaries F and F^-. We want to integrate the form $\xi \wedge \omega_F$ over the chain Γ', i.e. we want to calculate $\int_{\Gamma'} \xi \wedge \omega_F$, which equals

$$\int_{\Gamma'} \xi \wedge \omega_F = \int_{\delta_*(F\times\Gamma)} \xi \wedge \omega_F = \int_{\delta_*(F\times\iota(\Gamma))} \xi \wedge \omega_F,$$

since $id \times \iota : F \times F \to F \times F$ is an isomorphism. In view of example 2.6 and the Künneth formula, this is equal to

$$\frac{1}{2}\int_F (\phi^{P_0})^*\omega_F \cdot \int_\Gamma (\phi^{P_0})^*\xi.$$

The chain Γ', viewed as a singular chain on $J(C_0)$ has boundary

$$(\{0\} \times F) - (\{0\} \times F^-) = W_1(C_0) - W_1^-(C_0),$$

so we find

$$\nu_0([C_0])(\eta) = \frac{1}{2}\int_{\Gamma'} \eta = \int_F (\phi^{P_0})^*\omega_F \cdot \int_\Gamma (\phi^{P_0})^*\xi.$$

We can arrange this expression to be zero. The first factor is already nonzero because $(\phi^{P_0})^*\omega_F$ is a Kähler form. For the second we note that since P_0 is not a Weierstrass point, the chain Γ is not a cycle. This implies that $\int_{P_0}^{\iota(P_0)}$ is not a period of F and since $H^{1,0}(J(F)) \cong H^0(F, \Omega_F)$, we can find $\xi \in H^{1,0}(J(F))$ such that

$$\int_\Gamma (\phi^{P_0})^*\xi \neq 0.$$

This contradicts our conclusion from *step 1*, that the primitive normal function is zero.

Step 3. We argue with induction on the genus g. Suppose that the theorem holds up to $g-1$. Let $C_0 = E+X$ be stable curve of genus g, with E an elliptic curve and X a generic nonsingular complete curve of genus $g-1$. For the Jacobian varieties we have $J(C_0) = J(X) \times J(E)$. The r-cycle W_r specializes to

$$W_r(X) \times \{0\} + W_{r-1}(X) \times E. \tag{3.2}$$

If we assume the Ceresa r-cycles to be algebraically equivalent to zero, then

$$(W_r(X) \times \{0\} + W_{r-1}(X) \times E) - \big(W_r^-(X) \times \{0\} + W_{r-1}^-(X) \times E\big) \sim_{alg} 0 \tag{3.3}$$

Projecting onto $J(F)$ yields

$$W_r(X) - W_r^-(X) \sim_{alg} 0 \quad \text{on } J(X), \quad r = 0, 1, \ldots, g-1.$$

This contradicts the hypothesis for $r = 1, \ldots, g-3$. For the Ceresa $(g-2)$-cycle, note that the equations (3.2) and (3.3) imply that

$$(W_{g-3}(X) \times E) - \big(W_{g-3}^-(X) \times E\big) \sim_{alg} 0.$$

Intersecting with the cycle $J(X) \times \{0\}$ and projecting onto $J(X)$ yields

$$W_{g-3}(X) - W_{g-3}^-(X) \sim_{alg} 0,$$

again contradicting the hypothesis. This finishes the final step and concludes the proof of the theorem of Ceresa. □

Remark 3.2 We would like to stress that the original proof by Ceresa was not in the language of the normal functions. By using these functions anyway, we tried to clarify what happens on a conceptual level.

3.2 B. Harris' papers

In his paper [?], Bruno Harris defines the concept of a harmonic volume. Let C be a nonsingular curve over $\mathbb{C}$ of genus g. The harmonic volume is an element $I(\xi_1, \xi_2, \xi_3)$ of the circle $\mathbb{R}/\mathbb{Z}$, associated to a triple (ξ_1, ξ_2, ξ_3) of real harmonic forms on C that satisfies the following two conditions.

$$\int_C \xi_i \wedge \xi_j = 0, \quad 0 \le i, j \le 3 \tag{3.4}$$

$$\int_\Gamma \xi_i \in \mathbb{Z}, \quad \text{for any 1-cycle } \Gamma \text{ on } C. \tag{3.5}$$

The first first equation shows that $\xi_i \wedge \xi_j$ is exact. This gives a 1-form η_{ij} with $d\eta_{ij} = \xi_i \wedge \xi_j$. If we fix a point $P \in C$ we can define the map

$$x_i : C \to \mathbb{R}/\mathbb{Z} : Q \mapsto \int_P^Q \xi_i.$$

Let Γ_3 be the Poincaré dual of ξ_3. These ingredients are used to define the harmonic volume:

$$I(\xi_1, \xi_2, \xi_3) := \int_{\Gamma_3} x_1\xi_1 - \eta_{12} \in \mathbb{R}/\mathbb{Z}.$$

This is an integral that could be calculated, at least for certain curves. We will now explain why this is called a "harmonic volume", by presenting another interpretation of this number. The functions $x_1 : C \to \mathbb{R}/\mathbb{Z}$ together give a map

$$x = (x_1, x_2, x_3) : C \to T^3 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} : Q \mapsto (x_1(Q), x_2(Q), x_3(Q)).$$

The image of this map is a 2-cycle in the 3-dimensional torus T^3, and Harris shows that this chain is the boundary of a singular 3-chain Γ'. The volume of this chain modulo $\mathbb{Z}$, agrees with $I(\xi_1, \xi_2, \xi_3)$.

These considerations seem rather ad hoc, but we will now show how they correspond to the normal function of the previous section. To begin with, the harmonic 1-forms that satisfy (3.4) are denoted with $\mathrm{Har}^1_{\mathbb{Z}}(C)$. This is a free abelian group of rank $2g$. The elements in the kernel of the map $\bigwedge^3 \mathrm{Har}^1_{\mathbb{Z}}(C) \to \mathrm{Har}^1_{\mathbb{Z}}(C)$, defined on generators by

$$\xi_1 \wedge \xi_2 \wedge \xi_3 \mapsto \left[\int_C \xi_1 \wedge \xi_2\right] \xi_3 + \left[\int_C \xi_2 \wedge \xi_3\right] \xi_1 + \left[\int_C \xi_3 \wedge \xi_1\right] \xi_2,$$

give triples for which the harmonic volume is defined. The next step is to identify $\mathrm{Har}^1_{\mathbb{Z}}(C)$ with $H^1(C, \mathbb{Z})$ and in turn with $H^1(J(C), \mathbb{Z})$. We may also identify $\bigwedge^3 H^1(J(C), \mathbb{Z})$ with $H^3(J(C), \mathbb{Z})$. In view of these identifications, the kernel we described above corresponds to the classes of the closed primitive 3-forms $P^3_{\mathbb{Z}}(J(C))$. Harris shows that the association of a harmonic volume to a triple (ξ_1, ξ_2, ξ_3) corresponds to a homomorphism

$$I : P^3_{\mathbb{Z}}(J(C)) \to \mathbb{R}/\mathbb{Z}.$$

Then it is shown that the elements in $\mathrm{Hom}(P^3_{\mathbb{Z}}(J(C)), \mathbb{R}/\mathbb{Z})$ correspond to points on the 2^{nd} primitive intermediate Jacobian of C. In fact, we have

$$\mathrm{Hom}(P^{2k-1}_{\mathbb{Z}}(J(C)), \mathbb{R}/\mathbb{Z}) \cong J^k(J(C))_{prim}.$$

We will now show that the element $2I$ corresponds to the primitive Abel-Jacobi image of the Ceresa 1-cycle. To this end, we describe $J(C)$ by $\mathrm{Hom}(\mathrm{Har}^1_{\mathbb{Z}}(J(C)), \mathbb{R}/\mathbb{Z})$. We use that $\mathrm{Har}^1_{\mathbb{Z}}(J(C)) = P^1_{\mathbb{Z}}(J(C))$. Then $\mathrm{Hom}(\mathrm{Har}^1_{\mathbb{Z}}(J(C)), \mathbb{R}/\mathbb{Z})$ is actually equal to $J^1(J(C))_{prim}$, but all the 1-forms are primitive. Moreover, $J^1(J(C))$ is canonically isomorphic to $J(C)$ by auto-duality[1]. If we fix a basis $\{\xi_1, \ldots, \xi_{2g}\}$ for

[1] This is the phenomenon that $\mathrm{Alb}(\mathrm{Pic}^0(X)) = \mathrm{Alb}(X)$ and $\mathrm{Pic}^0(\mathrm{Alb}(X)) = \mathrm{Pic}^0(X)$. See e.g. p331 of [?].

$\mathrm{Har}^1_{\mathbb{Z}}(C)$, the embedding $\phi^P : C \to J(C)$ corresponds to the smooth map

$$C \to \mathrm{Hom}_{\mathbb{R}}(\mathrm{Har}^1_{\mathbb{Z}}(J(C)), \mathbb{R}/\mathbb{Z}) : Q \mapsto x_1(Q) + \cdots + x_{2g}(Q).$$

Composing with the projection $\pi : J(C) \to T^3$ of the first three coordinates gives the map $x : C \to T^3$. Let Γ be the singular 3-chain whose boundary is the Ceresa 1-cycle in $J(C)$. Then

$$\partial(\pi_*(\Gamma)) = \pi_*(C - C^-) = \pi_*(C) - (-1)_*\pi_*(C),$$

where $(-1)_*$ is the map in homology induced by the inverse on T^3. But $\pi_*(C) = x(C) = \partial(\Gamma')$, for some 3-chain Γ' in T^3. We find that

$$\partial(\pi_*(\Gamma)) = x(C) - (-1)_*x(C) = \partial(\Gamma' - (-1)_*(\Gamma')),$$

showing that $\pi_*(\Gamma)$ and $\Gamma' - (-1)_*(\Gamma')$ are homologous. By abuse of notation, we write

$$\Psi^1_{J(C)}(C - C^-)(\xi_1 \wedge \xi_2 \wedge \xi_3) = \int_\Gamma \xi_1 \wedge \xi_2 \wedge \xi_3 \quad \text{modulo periods.}$$

and Harris shows that right hand side equals the volume of $\pi_*(\Gamma)$ modulo $\mathbb{Z}$, which in turn equals the volume of $\Gamma' - (-1)_*(\Gamma')$ modulo $\mathbb{Z}$. The homology map $(-1)_*$ commutes with taking the volume, so we see that the value of the primitive Abel-Jacobi image of the Ceresa cycle, acting on a primitive holomorphic 3-form corresponds to taking twice the harmonic volume of this 3-form.

Harris furthermore shows that the harmonic volume is independent of the choice of embedding $C \to J(C)$ and that it is a holomorphic section of the family of 2^{nd} primitive intermediate Jacobians over a covering of the moduli space. To be precise, he considers the Torelli space, the moduli space of isomorphism classes of curves together with a basis of the homology group $H_1(C, \mathbb{Z})$. This is a fine moduli space, so it admits a universal curve. Note that $2I$ being holomorphic follows readily from the identification $2I([C]) = \nu_0([C])$ and our considerations in the previous section. He also shows that the differential of ν_0 is non-zero at some points of the hyperelliptic locus of the Torelli space. This shows that ν_0 is not the zero section on any fine moduli space of curves of genus 3 with additional structure. This gives a new counterexample for *step 2* in the proof of Ceresa's theorem. One could say, that reasoning further on, this gives a new proof of Ceresa's theorem. But for the induction argument we have to get back to Ceresa's methods of specializing to a degenerate curve. Indeed, his observation is not useful if we want to prove Ceresa's theorem for $g \geq 4, r = 1$, since ν_0 need not be the the zero section in those cases.
In 1995, William Faucette generalized the harmonic volume to higher dimensions in his paper [**?**]. He presents formulae to calculate the primitive Abel-Jacobi image acting on primitive forms. His results also cannot be used to prove Ceresa's theorem either: the normal functions involved need not be zero either. Before we move on to another proof of Ceresa's theorem for $g \geq 3, r = 1$, we present an application of the harmonic volume.

3.2.1 The Fermat curve of genus 3

In his paper [**?**], Bruno Harris shows that the Ceresa 1-cycle is not algebraically equivalent to zero for the curve

$$X^4 + Y^4 = Z^4.$$

This curve has been studied a lot, so there is a lot of information about it that can be applied to do some calculations.

Harris writes a primitive $(3,0)$-form as a real harmonic triple that satisfies (3.4) and (3.5) and then calculates its harmonic volume. With the choices he makes, this boils down to calculating

$$\frac{2\int_0^1 \left[\int_0^x \frac{dt}{(1-t^4)^{1/2}}\right] \frac{dx}{(1-x^4)^{3/4}}}{\left[\int_0^1 \frac{dt}{(1-t^4)^{1/2}}\right] \cdot \left[\int_0^1 \frac{dx}{(1-x^4)^{3/4}}\right]}.$$

The value is not in $\mathbb{Z}$, from which it follows that the harmonic volume of this triple is not an integer. This translates to the statement that $\nu_0([C])$ is non-zero.

REMARK 3.3 The above two counterexamples by Harris, for *step 1* in Ceresa's theorem don't give a really new proof of the theorem. The techniques that Ceresa used to construct his counterexample (specializing to a degenarated curve) are also used in the induction argument. As we remarked earlier, the nonvanishing of the differential of ν_0 does not serve as a counterexample for higher genera, since ν_0 need not be zero then.

3.3 Collino and Pirola's paper.

In this subsection we present the proof that Collino and Pirola [?] gave of Ceresa's thereom for $g \geq 3, r = 1$. Just like in the proof by Ceresa, we shall show that the assumption that the general Ceresa 1-cycle is algebraically equivalent to zero leads to a contradiction. We will work with a nonsingular connected and projective curve over $\mathbb{C}$ of genus $g \geq 3$. Let ν_0 denote the section of $\mathscr{J}^{g-1}_{prim}$ associated to the Ceresa 1-cycle. Under our assumption, the only thing we know about ν_0 is that it is a section of $\mathscr{J}^{g-1}_{alg}$. Therefore the non-vanishing of $d\nu_0$ doesn't lead to a contradiction. We need to be more precise in our considerations: this is where the infinitesimal invariant comes into play. Other ingredients of the proof are: an algebraic view on deformations and an adjunction map. We will introduce these notions briefly.

Let C be a curve as above. If we vary C in the moduli space, then we get a deformation of the pair[1] $(J(C), \omega)$ by the induced map $\mathfrak{M}_g \to \mathfrak{A}_g$. This gives a diffential $H^1(C, \mathscr{T}_C) \to H^1(J(C), \mathscr{T}_{J(C)})$ which factors through $H^1(J(C), \mathscr{T}_{J(C)})_\omega$. The latter space may be identified with the space of symmetric[2] homomorphisms

$$\mathrm{Hom}^{(s)}(H^0(C, \Omega_C), H^1(C, \mathscr{O}_C)).$$

Using these identifications, we get a map

$$H^1(C, \mathscr{T}_C) \to \mathrm{Hom}^{(s)}(H^0(C, \Omega_C), H^1(C, \mathscr{O}_C)) : \xi \mapsto \widehat{\xi},$$

where $\widehat{\xi}$ means cupping with ξ. If C is not hyperelliptic, then this map is injective. This follows from Max Noether's theorem, see e.g. [?]. The kernel of $\widehat{\xi}$ is denoted by Ξ_ξ.

Suppose for a moment that C occurs as the fiber of a deformation $\mathcal{C} \to B$ over $0 \in B$. A tangent vector $v \in T_0B$ is uniquely determined by a morphism Spec $\mathbb{C}[\epsilon]/(\epsilon^2) \to B$, which we also denote by v. Let $\xi = \rho_0(v)$ be the Kodaira-Spencer class of this

[1] The class ω is the first Chern class of the canonical ample bundle on $J(C)$.

[2] Symmetric with respect to the pairing $(\sigma, \tau) = \int_C \sigma \wedge \tau$, where $\sigma \in H^0(C, \Omega_C)$ and $\tau \in H^1(C, \mathscr{O}_C)$.

vector. Let C_ξ be a "thickening" of C, defined by the fibered diagram

$$\begin{array}{ccc} C_\xi & \longrightarrow & C \\ \downarrow & & \downarrow \\ \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2) & \xrightarrow{v} & \text{Spec}\, \mathbb{C}. \end{array}$$

Formally, such a thickening $C_\xi \to \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$ is characterized by the property that the fiber over 0 is isomorphic to C. We can associate such a thickening to every infinitesimal deformation and in fact: the isomorphism classes of these thickenings are paramatrized by $H^1(C, \mathscr{T}_C)$.

This way of looking at the infinitesimal deformations of C gives rise to an adjunction map that we construct now. Let ξ be an infinitesimal deformation of C and let C_ξ the associated thickening. The embedding $C \to C_\xi$ gives rise to the normal sheaf $\mathscr{N}_{C/C_\xi}$ and the restricted cotangent sheaf $\Omega_{C_\xi} \otimes \mathscr{O}_C$. We denote them briefly by $\mathscr{N}_\xi$ and Ω_ξ. They fit into the exact sequence

$$0 \to \mathscr{N}_\xi^* \to \Omega_\xi \to \Omega_C \to 0, \tag{3.6}$$

where the asterisk denotes the dual sheaf. The conormal sheaf is trivial, i.e. we have an isomorphism $\mu : \mathscr{N}_\xi^* \to \mathscr{O}_C$. Using this identification, we get the long exact sequence from (3.6),

$$0 \to H^0(C, \mathscr{O}_C) \to H^0(C, \Omega_\xi) \to H^0(C, \Omega_C) \xrightarrow{\partial} H^1(C, \mathscr{O}_C) \to \cdots$$

The first connecting homomorphism is equal to $\widehat{\xi}$. Taking the determinants of (3.6) gives an isomorphism $\bigwedge^2 \Omega_\xi \cong \mathscr{N}_\xi^* \otimes \Omega_C$. This shows that $\bigwedge^2 \Omega_\xi \otimes \mathscr{N}_\xi \cong \Omega_C$. We use the isomorphism $\mu^* : \mathscr{O}_C \to \mathscr{N}_\xi$ to get the map $H^0(C, \bigwedge^2 \Omega_\xi) \to H^0(C, \bigwedge^2 \Omega_\xi \otimes \mathscr{N}_\xi)$. Precomposing with the natural map $\bigwedge^2 H^0(C, \Omega_\xi) \to H^0(C, \bigwedge^2 \Omega_\xi)$ yields the adjunction map

$$\alpha_\xi : \bigwedge^2 H^0(C, \Omega_\xi) \to H^0(C, \Omega_C).$$

We use this adjunction map to define a pairing

$$\mathcal{Q}_\xi : \bigwedge^2 \Xi_\xi \times H^0(C, \Omega_C) \to \mathbb{C}$$

as follows. Let s_1 and s_2 be in Ξ_ξ. They lift to sections σ_1 and σ_2 of $H^0(C, \Omega_\xi)$. Then we can calculate $\alpha_\xi(\sigma_1 \wedge \sigma_2)$. Define

$$\mathcal{Q}(\sigma_1 \wedge \sigma_2, \tau) := \int_C \alpha_\xi(\sigma_1 \wedge \sigma_2) \wedge \widehat{\xi}(\tau).$$

This definition is independent of the chosen liftings. For this pairing to be interesting at all, we want $\dim \Xi_\xi \geq 2$. Among the infinitesimal deformations with $\dim \Xi_\xi \geq 2$, we have the special ones, for which $\mathcal{Q}_\xi = 0$ and the general ones, for which $\mathcal{Q}_\xi \neq 0$. The first important result from the paper [?] is that every non-hyperelliptic curve has a general deformation.

The infinitesimal invariant of a normal funtion has been introduced by Griffiths in his paper [?]. In fact, he uses a whole section to define it. Here we will do with an informal description. Let $\mathcal{C} \to B$ be a deformation of C. The infinitesimal invariant $\delta\nu_0$ of the normal function ν_0 from above is a global section of some (tautological) bundle on some (incidence) variety and it can be "evaluated at a point

$b \in B$". The resulting object is a linear map $K \to \mathbb{C}$, where K is the kernel of the map

$$T_bB \otimes P^{2,1}_{\mathbb{C}}(J(C_b)) \to P^{1,2}_{\mathbb{C}}(J(C_b)) : \sum_i \xi_i \otimes \eta_i \mapsto \sum_i \nabla_{\xi_i}(\eta_i). \tag{3.7}$$

Before we can describe this linear map, we need to give another description of the primitive intermediate Jacobian. Using Poincaré duality, we find

$$F^2P^3_{\mathbb{C}}(J(C_b))^* \cong \omega^{g-3}P^3_{\mathbb{C}}(J(C_b))/\omega^{g-3}F^1P^3_{\mathbb{C}}(J(C_b))$$

and accordingly, we find a (dual) lattice. The most important thing is that we can lift ν_0 to a section $\widetilde{\nu}_0$ of $\{\omega^{g-1}P^3_{\mathbb{C}}(J(C_b))\}_{b\in B}$. The infinitesimal invariant of ν_0, evaluated at b is defined by

$$\delta\nu_0(b)(\sum_i \xi_i \otimes \eta_i) = \sum_i \int_{J(C_b)} (\nabla_{\xi_i}\widetilde{\nu}_0) \wedge \eta_i.$$

If ν_0 is constant on B, then the $\delta\nu_0$ will be zero.

We will now see what happens if ν_0 is a section of $\mathscr{J}^{g-1}_{prim,alg}$. Let $\mathscr{L}ie(\mathscr{J}^{g-1}_{prim,alg})$ denote the sheaf of Lie-algebras on B of the family of abelian varieties

$$\{J^{g-1}(J(C_b))_{prim,alg}\}_{b\in B}.$$

Let $\{L_b\} \subset \{P^3_{\mathbb{C}}(J(C_b))\}_{b\in B}$ be the bundle of sections that are fiberwise orthogonal to $\mathscr{L}ie(\mathscr{J}^{g-1}_{prim,alg})$, i.e.

$$L = \{\eta \in P^3_{\mathbb{C}}(J(C)) | \int_C \eta \wedge \psi = 0, \forall \psi \in \mathrm{Lie}(J^{g-1}(J(C))_{prim,alg})\}$$

on each fiber, were we left out the subscript b. Note that $P^{3,0}_{\mathbb{C}}(J(C)) \subset L$. We denote this space with $L^{3,0}$. The space $L \cap P^{2,1}_{\mathbb{C}}(J(C))$ is denoted with $L^{2,1}$. Let K' denote the kernel of the map

$$T_bB \otimes L^{2,1} \to P^{1,2}_{\mathbb{C}}(J(C))$$

that is the restriction of (3.7). Clearly, $K' \subset K$. If $\xi \otimes \eta \in K'$, then $\int_C \nabla_\xi \widetilde{\nu}_0 \wedge \eta = 0$, since

$$\nabla(\mathscr{L}ie(\mathscr{J}^{g-1}_{prim,alg})) \subset \mathscr{L}ie(\mathscr{J}^{g-1}_{prim,alg}) \otimes \Omega_B.$$

This shows that if we assume the general Ceresa 1-cycle to be algebraically equivalent to zero, then $\delta\nu_0(b)(K'_b) = 0$. This observation can be compared to the observation in *step 1* of Ceresa's proof. The difference is that we get a conclusion that holds for all $g \geq 3$.

We will now lead the above conclusion to a contradiction. Collino and Pirola came up with the following formula of $\delta\nu_0(b)$, using the adjunction map we discussed above. Suppose that $\dim \Xi_\xi \geq 2$. Let $\sigma_1, \sigma_2 \in \Xi_\xi$ be linearly independent, and $\tau \in H^0(C, \Omega_C)$. Then we have

$$\delta\nu_0(0)(\xi \otimes \sigma_1 \wedge \sigma_2 \wedge \widehat{\xi}(\tau)) = -2\mathcal{Q}_\xi(\sigma_1 \wedge \sigma_2, \tau).$$

If C is not hyperelliptic, we can always find a general deformation ξ, with $\sigma_1, \sigma_2 \in \Xi_\xi$ and τ such that $\mathcal{Q}_\xi(\sigma_1 \wedge \sigma_2, \tau) \neq 0$. It remains to show that $\xi \otimes \sigma_1 \wedge \sigma_2 \wedge \widehat{\xi}(\tau)$ is in K'. To this end, Collino and Pirola first show that it is in K. Then they note that

$\sigma_1 \wedge \sigma_2 \wedge \tau$ is a $(3,0)$-form, and hence it is in L. Application of ∇_ξ and projection onto $L^{2,1}$ gives $\sigma_1 \wedge \sigma_2 \wedge \widehat{\xi}(\tau) \in L^{2,1}$ and thus

$$\xi \otimes \sigma_1 \wedge \sigma_2 \wedge \widehat{\xi}(\tau) \in K'.$$

Summing up, if we assume that the general Ceresa 1-cycle is algebraically equivalent to zero then ν_0 is a section of $\mathscr{J}^{g-1}_{prim,alg}$ over some fine moduli space of curves of genus g. In particular, the infinitesimal invariant, evaluated at a point and restricted to the space L, is trivial. However, outside the hyperelliptic locus this is shown to be false, since we can always find a general deformation in this case. □

As Collino and Pirola remarked in their paper, it would be interesting to compare their results to those of Harris. For example, does Harris use the infinitesimal invariant in disguise? In this case, what do his results imply in view of Collino and Pirola's approach?

The results by Collino and Pirola have been generalized by Atsushi Ikeda in his article [**?**]. His result provides an algebraic proof of Ceresa's theorem. It is perhaps interesting to compare his result to that of Faucette in [**?**]. This comparison will give insight in how the results of Collino and Pirola are related to that of Harris.

4 Tautological ring

Let C be a nonsingular connected projective curve. Its embedding $\phi^P : C \to J(C)$ gives classes C in the rational homology ring and the algebraic equivalence ring (with extended scalars) of $J(C)$. Let $\mathbf{H}$ denote the either of these rings for a moment. Since $J(C)$ is abelian, we have induced homomorphisms

$$[k]^*, [k]_* : \mathbf{H} \to \mathbf{H}.$$

Moreover, there is a second product besides the intersection product, that makes $\mathbf{H}$ into a commutative group. This is the Pontryagin product $\star$ that we defined earlier (section 1.3). The class C in $\mathbf{H}$ gives rise to new classes, by application of $\star, \cdot, [k]^*$ and $[k]_*$. All the classes we obtain in this way are called tautological classes. They are contained in the smallest $\mathbb{Q}$-vector subspace that contains the class C of the curve, and that is stable under the natural operations $\star, \cdot, [k]^*$ and $[k]_*$. This vector space is called the tautological ring and we denote it with $\mathcal{R}$.
In case $\mathbf{H}$ is the homology ring, we have the Poincaré formula. Using this formula we show that the ring $\mathcal{R}$ is just $\mathbb{Q}[\Theta]/(\Theta^{g+1})$, where Θ denotes the homology class of W_{g-1}. This is explained in subsection 4.1
In case $\mathbf{H}$ is the algebraic equivalence ring of a generic curve, we do not have this formula at our disposal. This follows from Ceresa's theorem. Moreover, it has been shown that $\mathcal{A}_{\mathbb{Q}}(J(C))$ is infinite dimensional for a generic curve of genus 3. So it is not even obvious that $\mathcal{R}$ is finite dimensional. But surprisingly it is, as has been shown by Arnaud Beauville in [**?**]. We sketch the proof of this result in subsection 4.2.
After we have determined the generators of $\mathcal{R}$, we are going to collect relations. We start with the vanishing of generators, in case the curve admits a certain covering of $\mathbb{P}^1$. This is the famous result by Colombo and Van Geemen (see [**?**]). This result has been generalized by Herbaut in [**?**]. Polishchuk [**?**] found relations for the generators of $\mathcal{R}$ that hold for every curve of fixed genus.

4.1 Poincaré's formula in the homology ring

Let C be a curve of genus g as above. Consider the integral homology ring of the Jacobian variety of C,

$$H_\bullet(J(C), \mathbb{Q}) = \bigoplus_{k=0}^{2g} H_k(J(C), \mathbb{Q}).$$

Poincaré dual to this ring is the cohomology ring,

$$H^\bullet(J(C), \mathbb{Q}) = \bigoplus_{k=0}^{2g} H^k(J(C), \mathbb{Q}).$$

The Poincaré duality map $PD : H_k(J(C), \mathbb{Q}) \to H^{2g-k}(J(C), \mathbb{Q})$ provides an isomorphism between these rings, changing the intersection product into the wedge product. The homology classes of the subvarieties $W_1, W_2, \ldots, W_{g-1}$ of $J(C)$ are denoted with the same symbols, except for W_1 and W_{g-1}. These are denoted with C and Θ. The Poincaré duals are denoted with

$$\eta_C, \eta_{W_2}, \ldots, \eta_\Theta.$$

Note that η_Θ is the first Chern class of the divisor Θ. We may view this class as a Kähler class on $J(C)$, and from this it follows that $\wedge^g \eta_\Theta \neq 0$. We can say more about the powers of η_Θ.

Theorem 4.1 (Poincaré) With the notation from above, we have

$$\eta_{W_n} = \frac{1}{(g-n)!} \bigwedge^{g-n} \eta_\Theta.$$

Proof. See [?] or [?]. □

This translates to a dual statement in homology: for $n = 1, \ldots, g$, the class of the n-fold self-intersection of the divisor Θ equals the class W_{g-n} times $n!$.
Consider the isogeny $[k] : J(C) \to J(C)$. We want to see what the induced maps $[k]_*$ and $[k]^*$ in homology are. The first one is just the push-forward. For the second we take the class of the inverse image of the cycle by $[k]$. To see what these classes are, we transfer to cohomology. The differential of $[k]$ is just multiplication with k, so $[k]^*$ acts on $H^1(J(C), \mathbb{Q})$ by multiplication with k. Since we have

$$H^p(J(C), \mathbb{Q}) \cong \bigwedge^p H^1(J(C), \mathbb{Q}),$$

the homomorphism $[k]^*$ is just multiplication with k^p. The push-forward of a differential n-form ξ by $[k]$ is defined as follows. The map $[k]$ is an étale morphism of degree k^{2g}, so the preimage of a small open set is the union of k^{2g} open subsets of $J(C)$. We can lift an n-tuple of vector fields to these slices, and sum up the values of ξ on these n-tuples. For a cohomology class ξ this shows that

$$[k]_*\xi = k^{2g-n}\xi.$$

In view of the projection formula $[k]_*([k]^*(\xi) \wedge \eta) = \xi \wedge [k]_*(\eta)$ with $\eta = 1$, we get $[k]_*[k]^*\xi = k^{2g}\xi$. Summing up, for $\xi \in H^n(J(C), \mathbb{Q})$ we have

$$[k]_*[k]^*\xi = k^{2g}\xi, \quad [k]^*\xi = k^{2n}\xi \quad \text{and} \quad [k]_*\xi = k^{2g-2n}\xi.$$

The Poincaré duality map is compatible with the natural actions of $[k]$ on homology and cohomology (by natural we mean the push-forward and pull-back in homology and cohomology respectively), i.e. the following diagram commutes.

$$\begin{array}{ccc} H_\bullet(J(C), \mathbb{Q}) & \xrightarrow{PD} & H^\bullet(J(C), \mathbb{Q}) \\ {\scriptstyle [k]_*}\big\downarrow & & \big\uparrow{\scriptstyle [k]^*} \\ H_\bullet(J(C), \mathbb{Q}) & \xrightarrow[PD]{} & H^\bullet(J(C), \mathbb{Q}) \end{array}$$

From this we deduce the actions of $[k]$ in homology. For $\alpha \in H_n(J(C), \mathbb{Q})$ we get

$$[k]_*[k]^*\alpha = k^{2g}\alpha, \quad [k]^*\alpha = k^{4g-2n}\alpha \quad \text{and} \quad [k]_*\alpha = k^{2n-2g}\alpha.$$

Finally, we want to see how the Pontryagin product transfers to cohomology. Let

$$* : H^\bullet(J(C), \mathbb{Q}) \to H^\bullet(J(C), \mathbb{Q})$$

be the Hodge star operator with respect to the form η_Θ. Let

$$\widetilde{PD} : H_\bullet(J(C), \mathbb{Q}) \to H^\bullet(J(C), \mathbb{Q})$$

be the map that restricts to $(-1)^{g+n} * PD$ on the summands $H_n(J(C), \mathbb{Q})$.

Lemma 4.2 The map $\widetilde{PD} : H_\bullet(J(C), \mathbb{Q}) \to H^\bullet(J(C), \mathbb{Q})$ transfers the Pontryagin product into the wedge product, i.e., we have

$$\widetilde{PD}(\alpha \star \beta) = \widetilde{PD}(\alpha) \wedge \widetilde{PD}(\beta).$$

PROOF. See §4.10 of [?]. □

We can now reap the fruits of the above observations.

THEOREM 4.3 The tautological ring $\mathcal{R}$ in homology is isomorphic to $\mathbb{Q}[\Theta]/(\Theta^{g+1})$.

PROOF. The only thing that we have to show is that $\mathbb{Q}[\Theta]/(\Theta^{g+1})$ is stable under the Pontryagin product. Indeed, if this is the case then $\mathbb{Q}[\Theta]/(\Theta^{g+1})$ is contained in every space that is stable under the natural actions and that contains C. Hence it is contained in the smallest. To show that it is stable under $\star$, we transfer to cohomology. It is sufficient to show that

$$PD(\mathbb{Q}[\Theta]/(\Theta^{g+1})) = \mathbb{Q}[\eta_\Theta]/(\wedge^{g+1}\eta_\Theta)$$

is stable under the Hodge star operator $*$, since

$$PD(\alpha \star \beta) = (-1)^g *^{-1} (*PD(\alpha) \wedge *PD(\beta)) .$$

But $*\eta_\Theta = \wedge^{g-1}\eta_\Theta/(g-1)!$, which is in $\mathbb{Q}[\eta_\Theta]/(\wedge^{g+1}\eta_\Theta)$. □

Note that we have not put any constraints on the curve, except that is has to be nonsingular and projective. Due to the Poincaré formula, the tautological rings of the (co)homology rings of the Jacobian varieties are all alike. Even if the curve is special, e.g hyperelliptic. This is in contrast to the tautological ring of $\mathcal{A}_\mathbb{Q}(J(C))$. In this ring we do not have a Poincaré formula, for a generic curve. This is a corollary of Ceresa's theorem.

4.2 The tautological ring in the algebraic equivalence ring

Let C again be a nonsingular projective curve of genus g. The classes of the subvarieties $W_1, W_2, \ldots, W_{g-1}$ in $\mathcal{A}_\mathbb{Q}(J(C))$ are denoted with

$$C, w^{g-2}, \ldots, w^2, \theta$$

respectively. Suppose that we have a Poincaré formula, i.e.

$$w^k = \frac{1}{k!}\theta^k, \quad \text{for } k = 0, \ldots, g,$$

where the product on the right hand side is the intersection product. Then we find in particular that

$$[-1]^* w^2 = [-1]^* \left(\frac{1}{2}\theta^2\right) = \frac{1}{2}\theta^2 = w^2,$$

where the last equality follows from the fact that $\theta^2 \in \mathcal{A}_0^2(J(C))$. But this violates Ceresa's theorem for a generic curve of genus $g \geq 3$.

For a generic curve, there is no Poincaré formula in the algebraic equivalence ring $\mathcal{A}_\mathbb{Q}(J(C))$.

This observation inspires to investigate the tautological ring $\mathcal{R} \subset \mathcal{A}_Q(J(C))$. It will not have a simple description like the tautological ring in cohomology, but the advantage is that it carries geometric information about the curve. Arnaud Beauville determined the generators of $\mathcal{R}$. The following theorem is the content of his paper [?].

Theorem 4.4 (A. Beauville, 2004) Let C be a nonsingular projective curve over $\mathbb{C}$ of genus g. The tautological ring, as described above, is generated as a $\mathbb{Q}$-algebra by the classes $\theta, w^2, \dots, w^{g-2}, C$.

Sketch of proof. Consider the $\mathbb{Q}$-algebra $\mathcal{R}'$ that is generated by

$$\theta, w^2, \dots, w^{g-2}, C,$$

and where the product is the intersection product. If we can show it is stable under the actions of $[k]_*$ and $[k]^*$ and closed under the Pontryagin product, then $\mathcal{R}' = \mathcal{R}$. We will first show that $\mathcal{R}'$ is stable under $[k]_*$ and $[k]^*$. To this end, we give different generators for $\mathcal{R}'$. Let $\mathcal{R}'$ be as above and consider the equation

$$t^g - w^1 t^{g-1} + \dots + (-1)^g w^g$$

in $\mathcal{A}_{\mathbb{Q}}(J(C))[t]$. This factors in some extension ring,

$$t^g - w^1 t^{g-1} + \dots + (-1)^g w^g = \prod_{i=1}^{g} (t - \lambda_i). \tag{4.1}$$

We use the λ_i to produce the Newton polynomials

$$N_k = \frac{1}{k!} \sum_{i=1}^{g} \lambda_i^k$$

which are in $\mathcal{A}_{\mathbb{Q}}(J(C))$ again. For example, we have

$$N_1 = \theta, \quad N_2 = \frac{1}{2}\theta^2 - w^2, \quad N_3 = \frac{1}{6}\theta^3 - \frac{1}{2}\theta \cdot w^2 + \frac{1}{2} w^3, \dots$$

In fact, the class N_k is a polynomial in $w^1, \dots, w^g$ and conversely, w^k is a polynomial in $N_1, \dots, N_k$.
For a Jacobian variety $J(C)$ we have a special Fourier transform. The principal polarization gives an isomorphism $\varphi_\Theta : J(C) \mapsto J(C)^t$. This induces isomorphisms for the Chow rings and the algebraic equivalence rings,

$$\mathrm{CH}_{\mathbb{Q}}(J(C)) = \mathrm{CH}_{\mathbb{Q}}(J(C)^t) \quad \text{and} \quad \mathcal{A}_{\mathbb{Q}}(J(C)) = \mathcal{A}_{\mathbb{Q}}(J(C)^t).$$

We can describe the Fourier map as follows. The variety $J(C) \times J(C)$ carries a line bundle

$$\mathscr{L}'(\Theta) := m^* \mathscr{O}(\Theta) \otimes p_1^* \mathscr{O}(\Theta)^{-1} \otimes p_2^* \mathscr{O}(\Theta)^{-1},$$

where m is the multiplication on $J(C)$ and the p_i are the projections from $J(C) \times J(C)$ onto the factors and $\mathscr{O}(\Theta)$ is the line bundle associated to Θ. The map

$$1 \times \varphi_\Theta : J(C) \times J(C) \to J(C) \times J(C)^t$$

induces an isomorphism of the pairs

$$(J(C) \times J(C), \mathscr{L}'(\Theta)) \to (J(C) \times J(C)^t, \mathscr{P})$$

where $\mathscr{P}$ is the Poincaré bundle on $J(C) \times J(C)^t$. We denote the algebraic equivalence class of $\mathscr{L}'(\Theta)$ with $\ell \in \mathcal{A}^1(J(C) \times J(C))$. The Fourier transform is defined by $\mathcal{F}(x) = p_{2*}(e^\ell \cdot p_1^* x)$ and it satisfies the properties F1-F4 from the first section. An application of the Grothendieck-Riemann-Roch formula yields the equality

$$-\mathcal{F}(C) = N_1 + N_2 + \dots + N_g.$$

With regard to the Beauville decomposition $C = C_{(0)} + C_{(1)} + \cdots + C_{(g-1)}$, this shows that

$$-\mathcal{F}(C_{(k-1)}) = N_k \quad \text{and} \quad \mathcal{F}(N_k) = (-1)^{g+k} C_{(k-1)}.$$

Summing up, $\mathcal{R}'$ can be generated by elements $N_1, \ldots, N_g$ and since $N_k \in \mathcal{A}^k_{k-1}(J(C))$, we see that $\mathcal{R}'$, viewed as a $\mathbb{Q}$-vector subspace, is stable under the intersections product and the actions $[k]_*$ and $[k]^*$.
It remains to show that $\mathcal{R}'$ is closed under the Pontryagin product. For this it suffices to show that $\mathcal{R}'$ is stable under $\mathcal{F}$. Indeed, in that case

$$x \star y = (-)^g \mathcal{F}(\mathcal{F}(x) \cdot \mathcal{F}(y)) \in \mathcal{R}'.$$

The inclusion $\mathcal{F}\mathcal{R}' \subset \mathcal{R}'$ follows, as soon as we have shown that $\mathcal{F}\mathcal{R}'$ is stable under $\mathcal{F}$, i.e. if $\mathcal{F}\mathcal{F}\mathcal{R}' \subset \mathcal{F}\mathcal{R}'$. Note that $\mathcal{F}\mathcal{R}'$ is closed under the Pontryagin product. We first show that 1 and θ are in $\mathcal{F}\mathcal{R}'$.
The element $1 \in \mathcal{A}_{\mathbb{Q}}(J(C))$ is actually the class of w^0, or rather, of $J(C)$ itself. Since

$$[k]_* w^0 = k^{2g} w^0,$$

we have $1 = w^0 \in \mathcal{A}^0_0(J(C))$. For the class w^g, which may be represented by the class of the identity element of $J(C)$, we have $[k]_* w^g = k^{2g} w^g$, so $w^g \in \mathcal{A}^g_g(J(C))$. In fact, since $\mathcal{F} : \mathcal{A}^g_g(J(C)) \cong \mathcal{A}^0_0(J(C))$, we see that $\mathcal{F}(w^g)$ is a $\mathbb{Q}$-multiple of w^0. This shows that $1 \in \mathcal{F}\mathcal{R}'$.
To show that θ is in $\mathcal{F}\mathcal{R}'$, we use the equality $\mathcal{F}(C_{(0)}) = N_1 = \theta$. The Beauville decomposition of the curve gives the formula

$$[k]_* C = k^2 \left(C_{(0)} + k \cdot C_{(1)} + \cdots + k^{g-1} \cdot C_{(g-1)} \right).$$

Consider the subspace of $\mathcal{R}'$ that is generated by $C, [2]_*C, \ldots, [g]_*C$. This is clearly a subspace of the vector space generated by $C_{(0)}, \ldots, C_{(g-1)}$. In fact, these spaces are the same. The base change is performed by a nonsingular matrix[1]. This shows that $C_{(0)}$ is in $\mathcal{R}'$ which establishes the claim that $1, \theta \in \mathcal{F}\mathcal{R}'$. It follows that $e^\theta \in \mathcal{F}\mathcal{R}'$.
Next, Beauville shows that $\mathcal{F}\mathcal{R}'$ is stable under intersecting with θ. In view of the formula

$$\mathcal{F}(x) = e^\theta \left(([-1]^*(x) \cdot e^\theta \star e^{-\theta}) \right), \tag{4.2}$$

which is proven in [**?**], this shows that $\mathcal{F}\mathcal{R}'$ is stable under $\mathcal{F}$. This closes the sketch of the proof. □

REMARK 4.5 In his article, Beauville does not mention explicitly that 1 and θ are in $\mathcal{F}\mathcal{R}'$, which is essential for the proof. This is the reason for adding a proof of this fact.

REMARK 4.6 The cycle class map $\mathrm{cl} : CH_{\mathbb{Q}}(J(C)) \to H_\bullet(J(C), \mathbb{Q})$ is compatible with both the intersection and the Pontryagin product. Furthermore, it is compatible with the actions of $[k]_*$ and $[k]^*$. This implies that $\mathrm{cl}(N_i) = 0$ for $i \neq g$. It follows that the tautological ring in homology is equal to $\mathbb{Q}[\Theta]/(\Theta^g)$, giving a different proof of theorem 4.3.

[1] This matrix is similar to the one we discussed on page 9.

4.3 Collecting relations

The tautological ring is bigraded, since $N_1, \dots, N_{g-1}$ are homogenous for both gradations of $\mathcal{A}_{\mathbb{Q}}(J(C))$. The map

$$\mathbb{Q}[N_1, N_2, \dots, N_{g-1}] \to \mathcal{R}. \tag{4.3}$$

is a surjective ring homomorphism. We are interested in the kernel of this map, denoted with K_g. If we know the kernel, we know exactly what $\mathcal{R}$ is. Unfortunately, we don't know K_g, but Alexander Polishchuk [?] has determined a number of relations among the classes $N_1, \dots, N_{g-1}$. The expression $N_1^{[d]}$ denotes the class $N_1^d/d!$ if $d \geq 0$ and 0 otherwise.

Theorem 4.7 (A.Polishchuk, 2005) For every $k \geq 0$, every $n_1, \dots, n_k$ such that $n_i \in \mathbb{Z}_{\geq 2}$, and every $d \in \mathbb{Z}$ such that $0 \leq d \leq k-1$, we have

$$\sum \binom{m-1}{d+m-k} b(I_1)\dots b(I_m) N_1^{[g-d-m+k-\sum_{i=1}^k n_i]} N_{d(I_1)} \dots N_{d(I_m)} = 0 \tag{4.4}$$

in $\mathcal{A}_{\mathbb{Q}}(J(C))$ and where the sum is over all partitions $[1,k] = I_1 \sqcup I_2 \sqcup \dots \sqcup I_m$ of the set $[1,k] = \{1, \dots, k\}$ into the disjoint union of nonempty subsets $I_1, \dots, I_m$, with $k - d \leq m \leq g - d + k - \sum_{i=1}^k n_i$ (two partitions differing only by an ordering of parts are considered to be the same); for a subset $I = \{i_1, \dots, i_s\} \subset [1,k]$ we denote

$$\begin{aligned} b(I) &= \tfrac{(n_{i_1}+\dots+n_{i_s})!}{n_{i_1}!\dots n_{i_s}!}, \\ d(I) &= n_{i_1} + \dots + n_{i_s} - s + 1. \end{aligned}$$

Proof. See [?]. □

Polishchuk shows on top of this, that the $\mathbb{Q}$-linear span of all the elements in $\mathcal{A}_{\mathbb{Q}}(J(C))$ that satisfy (4.4) is an ideal in $\mathbb{Q}[N_1, \dots, N_{g-1}]$. It is clearly contained in the kernel of (4.3) and it is conjectured that for a generic curve, this is an equality. Polishchuk gives a motivation for this conjecture in [?].

As a corollary, Polishchuk shows that $N_n = 0$ for $n \geq \frac{g}{2} + 1$. Note that a generic curve of genus g has a base point free g_d^1, for some $d \leq \frac{g+3}{2}$ (see e.g. Ch. V, Thm 1.1). In this way we recover the famous result by Elisabetta Colombo and Bert van Geemen for generic curves.

Theorem 4.8 (Colombo and Van Geemen, 1993) Let C be a curve of genus g and suppose it has a g_d^1. Then the classes $C_{(d')}$ are zero in $\mathcal{A}_Q(J(C))$, for $d' \geq d-1$. Equivalently, the classes $N_{d''}$ are zero for $d'' \geq d$.

Proof. See Thm. 1.3.3 of [?]. □

This is a vanishing result that leads to an effective description of $\mathcal{R}$ for hyperelliptic curves. Suppose that C is hyperelliptic of genus g. The ring $\mathcal{R}$ is generated by θ, so we need to determine the kernel of the projection

$$\mathbb{Q}[\theta] \to \mathcal{R}.$$

We claim that this is just (θ^{g+1}). We show that any power θ^k with $1 \leq k \leq g$ is nonzero. The Riemann-Roch theorem for abelian varieties states

$$\chi(\mathscr{O}(\Theta)) = \frac{\deg \Theta^g}{g!}$$

where $\chi(\mathscr{L})$ is the Euler characteristic

$$\chi(\mathscr{L}) = \sum_{i\geq 0}(-1)^i \dim H^i(J(C), \mathscr{L}).$$

But $\chi(\varphi_\Theta) = 1$, since φ_Θ is an isomorphism (for proofs of these facts, see e.g. [?], Thm. 13.3). We conclude that θ^g equals $g!$ times the class of a point on $J(C)$. This shows that no lower power θ^k is zero. Since $\theta^{g+1} = 0$, this establishes the claim. Still under the assumption that C is hyperelliptic, we take a closer look at the Newton polynomials. Expanding the right hand side of equation (4.1) and comparing coefficients yields

$$w^k = \frac{1}{k!}(N_1)^k \mod (N_2, \dots, N_g).$$

But the N_i are already zero for $i = 2, \dots, g$, since C is hyper-elliptic. It follows that algebraic equivalence ring of a hyper-elliptic curve has a Poincaré formula. This has been shown earlier by Alberto Collino in his paper [?]. We sketch his proof briefly.

Collino's proof of the Poincaré formula in $\mathcal{A}(J(C))$. Let C be a hyperelliptic curve of genus g and let $P \in C$ be a Weierstrass point. We introduce two subvarieties of $C^{(n)}$ for $1 \leq n \leq g$. The first one is $S(n)$; it paramatrizes the positive divisors on C of degree n whose complete linear system is non-trivial. More precise, with $\phi^P_{(n)} : C^{(n)} \to J(C)$ as usual,

$$S(n) := \{D \in C^{(n)} | \quad \dim |D| \geq 1\}.$$

It turns out that $S(n)$ is a divisor. The second variety is the image of $S(n)$ by the morphism $\phi^P_{(n)}$ and it is denoted by $G(n)$. It has been shown by Henrik Martens [?] that $G(n)$ is of dimension $n-2$ for $n = 1, \dots, g-1$ if and only if C is hyperelliptic. In particular, if we denote the class of $S(n)$ in $\mathrm{CH}(J(C))$ by s_n, then $(\phi^P_{(n)})_* s_n = 0$. The next observation is that the Chow ring of $C^{(n)}$ is a $\mathrm{CH}(J(C))$-algebra via $(\phi^P_{(n)})^*$. The image of

$$C^{(n-1)} \to C^{(n)} : D' \mapsto D' + P$$

gives a class $z \in \mathrm{CH}(C^{(n)})$, and this class generates $\mathrm{CH}(C^{(n)})$ as a $\mathrm{CH}(J(C))$-algebra. Collino gives a proof for the crucial identity

$$s_n = (\phi^P_{(n)})^*[W_{g-1}] - (g-n+1)z$$

in $\mathrm{CH}(C^{(n)})$. Pushing this forward to $\mathrm{CH}(J(C))$ gives

$$0 = [W_{g-1}] \cdot [W_n] - (g-n+1)[W_{n-1}].$$

Passing to the algebraic equivalence ring, we get $w^1 w^{g-n} = (g-n+1)w^{g-n+1}$. We apply induction to n to prove the Poincaré formula:

$$\frac{\theta^n}{n!} = w^n, \quad \text{for } n = 1, \dots, g.$$

in $\mathcal{A}_\mathbb{Q}(J(C))$. □

The Poincaré formula for hyperelliptic curves shows that $[-1]^*C = C$ in the algebraic equivalence ring. Moreover, if $C_{(1)} \neq 0$ in $\mathcal{A}_\mathbb{Q}(J(C))$, then the curve can't be hyperelliptic. This is an important observation, since it shows that the algebraic equivalence ring carries geometric information about the curve that we cannot reveal from the cohomology ring of the the Jacobian of the curve.

The result by Colombo and Van Geemen has been generalized by Fabien Herbaut. In his article [?] he presents a set of relations for the generators of $\mathcal{R}$, in case C has a base-point free g^r_d. His relations have been simplified by Gerard van de Geer and Alexis Kouvidakis in their article [?]. We present these simplified relations in a theorem.

THEOREM 4.9 (G. VAN DER GEER, A. KOUVIDAKIS, 2007) If C has a base point free g^r_d then

$$\sum_{a_1+\cdots+a_r=n}(a_1+1)!\dots(a_r+1)!N_{a_1+1}\dots N_{a_r+1}=0,$$

for $n \geq d-2r+1$. □

Note that for a d-gonal this yields $k!N_k=0$ for $k \geq d-1$. Again, we recover the result by Colombo and Van Geemen.
Herbaut showed another interesting thing. We state it in a theorem

THEOREM 4.10 Let C be a curve of genus g with a base point free g^r_d. Let $A(r,d,g)$ denote the number

$$\sum_{i=0}^{r-1}\frac{(-1)^i}{d-2r+2}\binom{i+g+r-d-2}{i}\binom{d-2r}{r-1-i}\binom{d-r+1-i}{r-i}.$$

If $A(r,d,g)\neq 0$ then $C_{(i)}=0$ for $i \geq d-2r+1$.

PROOF. Herbaut gives two proofs in [?]. We give a sketch of the second. First he proves that $A(r,d,g)$ denotes the number of $(r-2)$-planes $\mathbb{P}^{r-2}$ in $\mathbb{P}^r$ such that the intersection of the image of $C\to\mathbb{P}^r$ and $\mathbb{P}^{r-2}$ consists of $2r-2$ points, whenever this number is finite. Assuming $A(r,d,g)\neq 0$, we can consider such a $\mathbb{P}^{r-2}$ and the linear system of hyperplanes in $\mathbb{P}^r$ containing $\mathbb{P}^{r-2}$. This system is 1-dimensional, so it gives a g^1_{d-2r+2} on C. Application of Colombo and Van Geemen yields the result. □

We conclude this subsection with a question of Beauville. In his paper [?] he asked whether for a generic curve of genus ≥ 5, the class N_3 is zero or not. Let's consider this question here for $g=5$. We may disregard the hyperelliptic curves, since for those we have $N_2=N_3=0$. This allows us to identify C with its canonical embedding in $\mathbb{P}^4$. This is a curve of degree 8 and we denote is also with C. The canonical divisor on C is denoted with K.
Now C has a g^1_3 if and only if it has no g^3_7 (see e.g. Ex.6.4.6 of [?]). Theorem 4.10 with g^3_7 does not imply a vanishing of $C_{(2)}$. This can be seen in two ways. The first is to calculate $A(3,7,5)$; this number is 0. A more geometric reasoning is the following.
Let C be the curve in $\mathbb{P}^4$ and let $P\in C$. Projecting from P to $\mathbb{P}^3$ gives C', a nonsingular curve of degree 7. This has a trisecant. Indeed, if it hasn't, we could project onto $\mathbb{P}^2$ to obtain a nonsingular curve of degree 6. But such a curve has genus 10, which is absurd. Let $Q',R',S'\in C'$ be points on a line. Suppose that there is a fourth point T' on this line. Denote the inverse images on C by Q,R,S and T. These points span a $\mathbb{P}^2$ in $\mathbb{P}^4$ but no three of them are colinear. So the linear system of hyperplanes that contains the points P,Q,R,S and T is of dimension 1. This linear system equals $|K-P-Q-R-S-T|$, and application of Riemann-Roch yields

$$\dim|P+Q+R+S+T|=\dim|K-P-Q-R-S-T|+1=2.$$

This gives a g^2_5 on C. The image C'' of C is singular, having one node P''. The lines through P'' give a linear system on C of degree 3 and dimension 1, i.e. a g^1_3 on C. This is contradicting the assumption that C has a g^3_7, so there are no four points colinear on C'.
To answer the question of Beauville, we have to come up with other methods. Indeed, if a g^r_d on a generic curve implies $C_{(2)} = 0$ via theorem 4.10, then the locus of these curves is contained in the locus of curves with a g^1_3. The latter is of dimension ≤ 11, so the curve could not have geen generic in the first place.

4.4 Concluding remarks

Griffiths has shown in 1969 in [**?**] that algebraic and homological equivalence are not the same. Today we say that the Griffiths group

$$\mathrm{Griff}^k := \mathscr{Z}^k(X)_{hom}/\mathscr{Z}^k(X)_{alg}$$

for an algebraic variety X might be non-trivial. In 1974 it was not yet clear whether the Griffiths groups of an abelian variety were trivial or not. Indeed, Alberto Collino writes in the introduction of his paper [**?**]:

"...it is an open question whether they (the homological and algebraic equivalence) coincide on abelian varieties."

In fact, it was not even known at the time whether there was or wasn't a Poincaré formula on the Jacobian variety of a generic curve. In this thesis we have seen that both questions have been answered. The first is answered by Ceresa's theorem, in 1983. It gave rise to a new question: what does the tautological ring in the algebraic equivalence ring look like?
A first result directing to an answer was given by Colombo and Van Geemen in 1992. Beauville gave more structure to this ring in 1995 by providing the generators. Polishchuk contributed to a better understanding of the ring in 2005 by providing relations that hold for every curve. Herbaut and Van Der Geer and Kouvidakis finally contributed in 2007 with relations in the ring for particular curves.
Research to (the tautological ring of) the algebraic equivalence ring of a Jacobian variety is important because this ring is a finer invariant than the (tautological ring of) the cohomology ring. It may also lead to a better understanding of the Hodge conjecture on Jacobian varieties or to applications in number theory.
A better understanding may also give insight in the Bloch-Beilinson conjecture about filtrations on the Chow groups of smooth projective varieties. One can look for a discussion and references in the beautiful book [**?**] by Claire Voisin. For a survey article about algebraic cycles we refer to the interesting paper [**?**] by Kapil Hari Paranjape and Vasudevan Srinivas. It contains an overview of the subject together with 43 references.