THE BRANCHING RULES FOR $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$ REVISITED: A NOVEL PROOF AND APPLICATIONS

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Abstract. We study the branching rules for the pair $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$ by means of the extended weight semigroup. We obtain a new proof for the classical branching rules in this case as well as an approximation of the corresponding equivariant embeddings of the representation spaces. As an application we show that the approximations of the embeddings provide approximations of the corresponding spherical functions. In particular, this generates new examples of matrix weights for matrix valued orthogonal polynomials.

1. Introduction

Let G be a connected reductive group defined over $\mathbb{C}$ and let $H\subset G$ be a connected reductive subgroup. Let $\pi:G\to\mathrm{GL}(V)$ be an irreducible holomorphic representation. Since H is reductive, the restriction $\pi|_H:H\to\mathrm{GL}(V)$ decomposes in finitely many irreducible subrepresentations, i.e. $V=\bigoplus V'$ is a decomposition of V into a direct sum of irreducible H-subrepresentation spaces $V'\subset V$. Branching rules are concerned with describing which irreducible H-representations occur in this decomposition.

Suppose that some branching rule tells us that the irreducible H-representation $\pi':H\to\mathrm{GL}(V')$ occurs in the decomposition of the G-representation $\pi:G\to\mathrm{GL}(V)$. Two natural questions arise: (1) how often does π' occur in π, i.e. what is the multiplicity $[\pi|_H:\pi']:=\dim\mathrm{Hom}_H(V',V)$ of π' in π and (2) can we describe the H-equivariant embeddings $V'\to V$ explicitly?

1.1. **Statements of results.** In this note we study these questions for the pair $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$. We use the theory of spherical varieties to deduce the classical branching rules which are given by interlacing properties, see Theorems 2.1 and 3.1. The multiplicity question is trivial in this case because $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$ is a spherical pair, a notion that we explain below. Our description of the branching rules also gives rise to equivariant embeddings of ambient representation spaces that approximate the embeddings that we are interested in. This means that for two irreducible representations (π',V') and (π,V) of H and G respectively we find ambient representation spaces $S(H,V')$ and $S(G,V)$ with equivariant embeddings $V'\to S(H,V')$, $V\to S(G,V)$ and $S(H,V')\to S(G,V)$ such that the composition $V'\to S(H,V')\to S(G,V)\to V$ is injective, the last arrow being the equivariant projection.

As an application we obtain approximations of the corresponding spherical functions. In an example we calculate families of certain spherical functions explicitly, which in turn provides new examples of matrix weights for matrix valued orthogonal polynomials.

We proceed to discuss the content of this note in more detail, thereby introducing the necessary concepts and notations.

1.2. **Spherical varieties and representations.** A pair (G,P) with G a connected reductive group (always over $\mathbb{C}$ from now on) and $P\subset G$ a connected subgroup is called a spherical pair if a Borel subgroup $B\subset G$ has an open orbit in the quotient G/P. The group of characters $P\to\mathbb{C}^\times$ is denoted by $X(P)$. Let $\mu\in X(P)$ and consider the associated G-line bundle $G\times^P\mathbb{C}_\mu\to G/P$. Its space of global sections $\Gamma(G\times^P\mathbb{C}_\mu)$ is isomorphic to $\{f\in\mathbb{C}[G]|\forall(p,g)\in P\times G:\ f(gp)=\mu(p)^{-1}f(g)\}$. The group G acts on $\Gamma(G\times^P\mathbb{C}_\mu)$ by $g\cdot f(g')=f(g^{-1}g')$. The pair (G,P) is spherical if and only if for all $\mu\in X(P)$ the representation space $\Gamma(G\times^P\mathbb{C}_\mu)$ decomposes multiplicity free into irreducible G-representations, see e.g. [18, Thm.25.1].

Let (G,P) be a spherical pair. Fix a Borel subgroup $B\subset G$ with Levi decomposition $B=TU$, where $T\subset B$ is a maximal torus and U the maximal unipotent subgroup of B. Let $X^+(T)$ be the group of characters of T that are positive with respect to B. The irreducible G-representations are determined by their highest weight, i.e. if $\pi:G\to\mathrm{GL}(V)$ is an irreducible representation, then there is a unique line $\mathbb{C}v$ that is stable under B. The torus T acts on $\mathbb{C}v$ with a character called the highest weight of π. The vector $v\in V$ is called a highest weight vector and it is unique up to scaling. An irreducible representation of highest weight $\lambda\in X^+(T)$ is denoted by $\pi_\lambda:G\to\mathrm{GL}(V_\lambda)$ or simply by π_λ. We denote the highest weight of the dual representation V_λ^* by λ^*. We are interested in the set of pairs $(\lambda,\mu)\in X^+(T)\times X(P)$ such that

$$\mathbb{C}[G]^{(B\times P)}_{(\lambda,\mu)}:=\{f\in\mathbb{C}[G]|\,\forall(b,g,p)\in B\times G\times P:\ f(b^{-1}gp)=\lambda(b)\mu(p)f(g)\}$$

is non-trivial. The collection of all these pairs is called the extended weight semigroup,

$$\widehat{\Lambda}_+(G,P):=\{(\lambda,\mu)\in X^+(T)\times X(P)|\,\mathbb{C}[G]^{(B\times P)}_{(\lambda,\mu)}\neq\{0\}\}.$$

The extended weight semigroup has been studied for example in [2, 1]. Note that $(\lambda^*,\mu)\in\widehat{\Lambda}_+(G,P)$ if and only if $(V_\lambda)^{(P)}_\mu:=\{v\in V_\lambda|\,\forall p\in P:\ \pi_\lambda(p)v=\mu(p)v\}$ is non-trivial. Moreover, if G is simply connected, then $\widehat{\Lambda}_+(G,P)$ is freely generated by indecomposable elements, i.e. the generators are not multiples of other elements in $X^+(T)\times X(P)$. The reason is that these generators are the weights of the B-stable prime divisors on G/P. See e.g. [2, §1.2-3] for a discussion of these facts.

Let G be a connected reductive group, $H\subset G$ a connected spherical subgroup and $P\subset H$ a parabolic subgroup. The triple (G,H,P) is called a multiplicity free system if (G,P) is a spherical pair. The multiplicity free systems (G,H,P) have been classified in [7, Thm.6.2] and [20] for (G,H) symmetric and non-symmetric spherical respectively. If $\mu \in X^+(P)$, i.e. μ is the extension to P of a positive character $\mu \in X^+(T)$ that is trivial on the unipotent radical of P, then

$$\mathrm{ind}_P^H(-\mu) := \{f \in \mathbb{C}[H] \,|\, \forall p \in P : f(hp) = \mu(p)f(h)\}$$

is an irreducible H-representation of highest weight μ^*. Induction in stages shows that $\mathrm{ind}_P^G(-\mu) = \mathrm{ind}_H^G \mathrm{ind}_P^H(-\mu)$ and Frobenius reciprocity implies that $[\mathrm{ind}_P^G(-\mu) : \pi_\lambda] = 1$ if and only if $[\pi_\lambda|_H : \pi_{\mu^*}] = 1$ which is equivalent to $(\lambda^*, \mu) \in \widehat{\Lambda}_+(G,P)$. Hence a multiplicity free triple (G,H,P) provides an abundance of multiplicity free triples (G,H,π), i.e. irreducible representations π of H that induce multiplicity free to G.

Definition 1.1. *Let $P_G^+(\mu)$ denote the set of all $\lambda \in X^+(T)$ such that $[\pi_\lambda|_H : \pi_\mu] \geq 1$. The set $P_G^+(\mu)$ is called the μ-well.*

The set $P_G^+(\mu)$ describes the irreducible subrepresentations of $\mathrm{ind}_H^G \pi_\mu$. It is obtained from $\widehat{\Lambda}_+(G,P)$ by fixing the second coordinate to be μ and replacing the first coordinate by its dual.

The description of $P_G^+(\mu)$ is a generalization of the Cartan-Helgason theorem which describes the set $P_G^+(0)$ where $H \subset G$ is symmetric, see [9, Thm.8.49]. The sets $P_G^+(\mu)$ have been calculated for many examples of multiplicity free systems, see e.g. [4, 8, 20]. The calculations in these references are all based on the inversion of classical branching rules. Moreover, all these examples are of rank one, i.e. the multiplicity free systems (G,H,P) are such that (G,H) is a spherical pair of rank one. By the rank of a spherical variety we mean rank of the abelian subgroup of $X(T)$ that consists of weights of non-trivial semi-invariant rational functions on G/H. See e.g. [18, §5] for more details.

1.3. **Relation to matrix valued orthogonal polynomials.** The μ-well $P_G^+(\mu)$ has also been calculated for some multiplicity free systems of higher rank: in [17] for G/H symmetric and $\mu : H \to \mathbb{C}^\times$ a character and in [12, 20] for higher dimensional irreducible H-representations. In all these cases it turns out that the μ-well is of a particular shape $P_G^+(\mu) \cong B(\mu) \times \mathbb{N}_0^r$, where $B(\mu) \subset P_G^+(\mu)$ is a finite set and r is the rank of the spherical pair (G,H). This shape and additional properties of $B(\mu)$ allows to describe certain sets of matrix coefficients, namely the spherical functions Φ_λ^μ of type μ associated to $\lambda \in P_G^+(\mu)$, see Definition 5.1, by means of matrix valued orthogonal polynomials. For example, the zonal spherical functions on a symmetric space G/H, i.e. the spherical functions of type $0 \in P_H^+$, have the structure of a polynomial algebra with $\mathrm{rank}(G/H)$ generators. Being eigenfunctions to the Casimir operator, they can be written as hypergeometric functions. Since the parameters are integral, the hypergeometric series are only finite, which shows the polynomial behavior once more. We refer to [8, 12, 19, 20] for this connection with matrix valued orthogonal polynomials. The families of matrix valued orthogonal polynomials that are associated to the representation theory of spherical pairs can sometimes be made completely explicit, see e.g. [10, 11, 15].

The first examples of matrix valued orthogonal polynomials related to the representation theory of spherical pairs can be found in [13]. In [6] the representation theory for $(\mathrm{SL}(3,\mathbb{C}),\mathrm{GL}(2,\mathbb{C}))$ is studied by bringing invariant differential operators into the game. As a result of further studies the authors obtain families of matrix valued orthogonal polynomials in subsequent papers. One of the ingredients to see that the solutions of their differential equations are of polynomial nature, is the observation that the invariant differential operators can be brought into hypergeometric form. This implies that the solutions have a power series of hypergeometric nature on the one hand, but on the other hand the series is finite because the spectral parameter satisfies an integrality condition. The way to hypergeometrize the involved operators was first observed by Román and Tirao in [16] for the case $(\mathrm{SL}(3,\mathbb{C}),\mathrm{GL}(2,\mathbb{C}))$ and has later been modified for the more general pairs $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$ in [14]. However, that result holds only for step-one representations, which means that the highest weight is a multiple of a fundamental weight plus a multiple of the determinant representation.

In the general set-up [8, 12, 20] the link between the representation theory and the orthogonal polynomials is given by the set of spherical functions Φ^{μ}_{λ}, where $\lambda\in B(\mu)\subset P^{+}_{G}(\mu)$. It is clear from the theory that these functions give the necessary hypergeometrizations, although this is not used for an *a priori* construction of the polynomials. However, it is important to have control over the indicated set of spherical functions because they determine the matrix weight that describes the orthogonality of the polynomials. Using the approximations of the embeddings $V_\mu\to V_\lambda$ in this paper we provide approximations of the spherical functions. This means roughly that up to lower order and up to an invertible upper triangular matrix, we can describe packages of spherical functions by functions that we can calculate more easily.

This upper triangular matrix encodes a branching problem that is in general not multiplicity free. We give an explicit example of families where this matrix can be calculated. This provides a new family of matrix weights.

1.4. **Notation and conventions.** From this point on we fix $n\geq 2$ in $\mathbb{N}$ and throughout the rest of this paper the symbols G and H denote

$$G=\mathrm{SL}(n+1,\mathbb{C})\quad\text{and}\quad H=\mathrm{GL}(n,\mathbb{C}).$$

We view H as a subgroup of G via the embedding $h\mapsto\mathrm{diag}(h,\det(h)^{-1})$. Note that (G,H) is a spherical pair. Moreover, if $B_H\subset H$ is a Borel subgroup, then (G,B_H) is a spherical pair. For a reference see e.g. [7] or [19, note 2.2.14].

Let $B \subset G$ be the standard Borel subgroup consisting of upper triangular matrices. Let $T \subset B$ be the maximal torus consisting of diagonal elements. Let $\epsilon_i : T \to \mathbb{C}^\times : t = \operatorname{diag}(t_1, \ldots, t_{n+1}) \mapsto t_i$. We denote the characters additively. The set of roots of the pair (G, T) is denoted by $\Delta(G,T) = \{\pm(\epsilon_i - \epsilon_j) : 1 \le i < j \le n+1\}$. The set of positive roots is $\Delta^+(G,T) = \{\epsilon_i - \epsilon_j : 1 \le i < j \le n+1\}$. The set of simple roots is $\Pi(G,T) = \{\epsilon_i - \epsilon_{i+1} | \, i = 1, \ldots, n\}$. The Killing form is identified with the pairing $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. The fundamental weights are given by $\varpi_i = \sum_{j=1}^{i} \epsilon_j - j(\epsilon_1 + \cdots + \epsilon_{n+1})/(n+1)$ for $i = 1, \ldots, n$.

Let $B_H \subset H$ denote the standard Borel subgroup consisting of upper triangular matrices. The maximal torus $T \subset G$ is also maximal in H. Therefore we can describe the roots and weights of H in terms of $\epsilon_1, \ldots, \epsilon_{n+1}$. We have $\Delta(H,T) = \{\pm(\epsilon_i - \epsilon_j) : 1 \le i < j \le n\}$. The set of positive roots is $\Delta^+(G,T) = \{\epsilon_i - \epsilon_j : 1 \le i < j \le n\}$. The set of simple roots is $\Pi(H,T) = \{\epsilon_i - \epsilon_{i+1} | \, i = 1, \ldots, n-1\}$. The fundamental weights are $\varpi_1, \ldots, \varpi_{n-1}$. The character ϖ_n is the highest weight of the representation $H \to \mathbb{C}^\times : H \mapsto \det(h)$. Note that the positive Weyl chamber of H contains the positive Weyl chamber of G. We have visualized this in Figure 1.

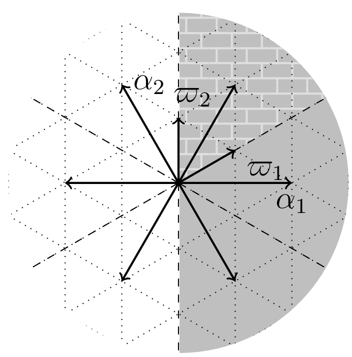

FIGURE 1. Roots and fundamental weights for $\mathrm{SL}(3,\mathbb{C})$. The positive Weyl chamber of $\mathrm{GL}(2,\mathbb{C})$ is light gray and it contains the positive Weyl chamber of $\mathrm{SL}(3,\mathbb{C})$ which is filled with bricks.

The pair (G, H) is a symmetric pair, because $H \subset G$ is the set of fixed points for the involutive automorphism $\theta : G \to G : g \mapsto I_{n,1} g I_{n,1}$, where $I_{n,1} = \operatorname{diag}(1, \ldots, 1, -1)$. Let $A \subset G$ denote the one-dimensional torus with elements

$$a_w := \begin{pmatrix} \frac{1}{2}(w + w^{-1}) & 0 & -\frac{1}{2}(w - w^{-1}) \\ 0 & I_n & 0 \\ \frac{1}{2}(w - w^{-1}) & 0 & \frac{1}{2}(w + w^{-1}) \end{pmatrix}, \quad w \in \mathbb{C}^\times.$$

Note that $\theta(a_w) = a_w^{-1}$. Let $H_* = Z_A(H)$, the centralizer of A in H. The elements of H_* are given by $\operatorname{diag}(z, y, z)$, where $z \in \mathbb{C}^\times, y \in \mathrm{GL}(n-1, \mathbb{C})$ and $z^2 \det(y) = 1$. Let $T_{H_*} \subset H_*$ denote the maximal torus consisting of diagonal elements. The roots, positive roots and fundamental weights of H_* are given by $\Delta(H_*, T_{H_*}) = \{\pm(\epsilon_i - \epsilon_j) | \, 2 \le i < j \le$

$n\}, \Delta^+(H_*, T_{H_*}) = \{\epsilon_i - \epsilon_j \,|\, 2 \leq i < j \leq n\}$ and $\{\widetilde{\varpi}_2, \ldots, \widetilde{\varpi}_n\}$, where $\widetilde{\varpi}_i = \varpi_i - \frac{1}{2}(\epsilon_1 - \epsilon_{n+1})$. The representation $H_* \to \mathbb{C}^\times : \mathrm{diag}(z, A, z) \mapsto z$ is of highest weight $\widetilde{\varpi}_1$.

We describe the irreducible representations of G, H and H_* by their highest weights. Let P_G^+, P_H^+ and $P_{H_*}^+$ denote the semigroups of dominant integral weights. We have

$$\begin{aligned} P_G^+ &= \mathbb{N}_0\varpi_1 \oplus \cdots \oplus \mathbb{N}_0\varpi_{n-1} \oplus \mathbb{N}_0\varpi_n, \\ P_H^+ &= \mathbb{N}_0\varpi_1 \oplus \cdots \oplus \mathbb{N}_0\varpi_{n-1} \oplus \mathbb{Z}\varpi_n, \\ P_{H_*}^+ &= \mathbb{Z}\widetilde{\varpi}_1 \oplus \mathbb{N}_0\widetilde{\varpi}_2 \oplus \cdots \oplus \mathbb{N}_0\widetilde{\varpi}_n. \end{aligned}$$

Let $U = \mathrm{SU}(n+1) \subset G$ be the set of unitary matrices, i.e. the elements $g \in G$ for which $\langle gv, gw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{C}^{n+1}$, where $\mathbb{C}^n$ is endowed the standard Hermitian inner product $\langle \cdot, \cdot \rangle$. Then $U \subset G, U \cap H \subset H$ and $U \cap H_* \subset H_*$ are maximal compact subgroups. We endow each representation space of G, H and H_* with a Hermitian inner product for which the actions of the indicated maximal compact subgroups are unitary. Whenever we say a map between representation spaces is isometric, it will be with respect to these Hermitian structures.

Finally we discuss the symmetric powers of representations. Let $\pi : G \to \mathrm{GL}(V)$ be a a representation of G. Let $v_1, \ldots, v_d$ be an orthonormal basis of weight vectors. Fix $k \in \mathbb{N}$. Let $S^k(V)$ denote the symmetric power of V. Viewed as a subspace of the k-th tensor product it inherits the canonical Hermitian structure. Let $\rho \in \mathbb{N}_0^d$ be a partition of k, i.e. $|\rho| := \sum_{i=1}^d \rho_i = k$. Denote by v_ρ the element $v_1^{\rho_1} \cdots v_d^{\rho_d} \in S^k(V)$. Denote by $\binom{k}{\rho} = k!/(\rho_1! \cdots \rho_d!)$ the multinomial of k and ρ. The elements $\binom{k}{\rho}^{-1/2} v_\rho$ with $\rho \in \mathbb{N}_0^d$ such that $|\rho| = k$ constitute an orthonormal basis of $S^k(V)$.

2. Inverting the branching rule for $\mathrm{SL}(n+1, \mathbb{C})$ to $\mathrm{GL}(n, \mathbb{C})$

The classical branching rules for the general linear groups $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(n+1, \mathbb{C})$ can be described by interlacing properties, see e.g. [9, Thm.9.14]. These rules have been proved by Weyl for the maximal compact subgroups $U(n) \subset U(n+1)$ and the irreducible representations of $U(n)$ correspond to the irreducible holomorphic representations of $\mathrm{GL}(n, \mathbb{C})$ via Weyl's unitary trick. The branching rules for $H \subset G$ can be deduced from these rules and they can be formulated as follows.

Theorem 2.1. *Let $\lambda \in P_G^+$ and $\mu \in P_H^+$ and write $\lambda = \sum_{i=1}^{n+1} a_i \epsilon_i$ and $\mu = \sum_{i=1}^{n+1} b_i \epsilon_i$. Then $[\pi_\lambda|_H : \pi_\mu] = 1$ if and only if* (i) *$a_i - b_i \in \mathbb{Z}$ and* (ii) *$a_i \geq b_i$ and $b_i \geq a_{i+1}$ for $1 \leq i \leq n$.*

We provide an alternative proof of Theorem 2.1 in Remark 2.4. Typically, in a branching rule we take a fixed irreducible G-representation and we describe which irreducible H-representations occur in it. Inversion of the branching rules means that we take a fixed irreducible H-representation and we describe which irreducible G-representations contain the given H-representation upon restriction to H. In principle Theorem 2.1 provides both the branching rule and its inversion. However, we want to recover more structure in the set of irreducible G-representations that contain a fixed irreducible H-representation. To this end we look at the extended weight semigroup.

Lemma 2.2. *The extended weight semigroup $\widehat{\Lambda}^+(G, B_H)$ is generated by*

$$(\omega_{n+1-i}, \omega_i), \quad (\omega_{n+1-i}, \omega_{i-1} - \omega_n), \quad i = 1, \ldots, n. \tag{1}$$

Proof. A B-stable prime divisor on G/P either maps onto one of the two B-stable prime divisors on G/H or it intersects the fiber H/B_H in a B_{H_*}-stable divisor. The Bruhat decomposition of H/B_H shows that there are $n-1$ prime divisors that are B_H-stable and thus B_{H_*}-stable. The open cell of H/B_H admits $n-1$ prime divisors that are B_{H_*}-stable. Indeed, these B_{H_*}-stable prime divisors correspond to the $T_{H_*} \cong (\mathbb{C}^\times)^{n-1}$-stable prime divisors on $\mathrm{Lie}(U_H)/\mathrm{Lie}(U_{H_*}) \cong \mathbb{C}^{n-1}$, the quotient ot the Lie algebras of the maximal unipotent subgroups of B_H and B_{H_*} respectively.

The elements (1) are linearly independent and indivisible. Furthermore it is clear that $(\varpi_{n+1-i}, \varpi_i) \in \widehat{\Lambda}^+(G, B_H)$, because $V^*_{\varpi_{n+1-i}} = V_{\varpi_i}$ has a B_H-stable line on which T_H acts with character ω_i. To see that the H-module $V_{\varpi_{i-1}-\varpi_n}$ is contained in V_{ϖ_i}, note the decomposition

$$\bigwedge^i \mathbb{C}^{n+1} = \bigwedge^i (\mathbb{C}^n \oplus \mathbb{C}_{-1}) = \left(\bigwedge^{i-1} \mathbb{C}^n \otimes \mathbb{C}_{-1} \right) \oplus \bigwedge^i \mathbb{C}^n \tag{2}$$

of H-representations. □

Proposition 2.3. *Let $\mu = \sum_{i=1}^n \mu_i \varpi_i$. An element $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in P_G^+(\mu)$ is uniquely determined by the pair of n-tuples $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ such that*

- $r_i + s_i = \lambda_i$ *for* $i = 1, \ldots, n$,
- $r_i + s_{i+1} = \mu_i$ *for* $i = 1, \ldots, n-1$,
- $r_n - (s_1 + \cdots + s_n) = \mu_n$.

Proof. For $\lambda \in P_G^+(\mu)$ we have $(\lambda^*, \mu) \in \widehat{\Lambda}^+(G, B_H)$, so by Lemma 2.2 there exists a unique pair of n-tuples $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ such that

$$(\lambda, \mu) = \sum_{i=1}^n \left(r_i(\varpi_i, \varpi_i) + s_i(\varpi_i, \varpi_{i-1} - \varpi_n) \right).$$

The result follows from comparison of the coefficients. □

Remark 2.4. We can now prove Theorem 2.1. Let $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in P_G^+$ and $\mu = \sum_{i=1}^n \mu_i \varpi_i \in P_H^+$. Then (λ, μ) satisfies the interlacing conditions of Theorem 2.1 if and only if there exists $s \in \mathbb{N}_0^n$ with (1) $s_i \leq \lambda_i$ for $i = 1, \ldots, n$ and (2) $\lambda - \mu = \sum_{i=1}^n s_i(\epsilon_i - \epsilon_{n+1})$.

Suppose that we are in this situation. Write $r_i = \lambda_i - s_i$ for $i = 1, \ldots, n$. Then $r \in \mathbb{N}_0^n$. We have to check that $\mu_i = r_i + s_{i+1}$ for $i = 1, \ldots, n-1$ and $\mu_n = r_n - (s_1 + \cdots + s_n)$. To this end write $\lambda = \sum_{i=1}^{n+1} a_i \epsilon_i$ and $\mu = \sum_{i=1}^{n+1} b_i \epsilon_i$. Observe that $s_i = a_i - b_i$ for $i = 1, \ldots, n$.

This implies that $r_i = b_i - a_{i+1}$ for $i = 1, \ldots, n-1$. Hence $r_i + s_{i+1} = b_i - b_{i+1} = \mu_i$ for $i = 1, \ldots, n-1$. Finally we use $\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} b_i = 0$ to deduce $r_n - (s_1 + \ldots + s_n) = \mu_n$.

Given a pair $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$, we define (λ, μ) by the conditions of Proposition 2.3. We have to show that the interlacing conditions of Theorem 2.1 are satisfied. This follows from the observation $\lambda - \mu = \sum_{i=1}^n s_i(\epsilon_i - \epsilon_{n+1})$.

This shows that Theorem 2.1 is equivalent to Proposition 2.3, so we obtain an alternative proof of the classical branching rules using the extended weight semigroup.

Note that $P_G^+(\mu)$ is stable under addition of multiples of $\varpi_1+\varpi_n = \alpha_1+\cdots+\alpha_n = \epsilon_1-\epsilon_{n+1}$. Moreover, given $\lambda \in P_G^+(\mu)$ we can subtract $r(\varpi_1 + \varpi_n)$ for $r = 0, \ldots, \min(r_n, s_1)$ without leaving the μ-well, i.e. $\lambda - r(\varpi_1 + \varpi_n) \in P_G^+(\mu)$ for all these r. Define

$$B(\mu) := \{\lambda \in P_G^+(\mu) : \lambda - (\varpi_1 + \varpi_n) \notin P_G^+(\mu)\},$$

the *bottom* of the μ-well $P_G^+(\mu)$. We have

$$P_G^+(\mu) = B(\mu) + \mathbb{N}_0(\varpi_1 + \varpi_n). \tag{3}$$

For a fixed $\mu \in P_H^+$ we define the μ-degree of $\lambda \in P_G^+(\mu)$ by $d_\mu(\lambda) := \min(r_n, s_1)$.

Lemma 2.5. *Let $\lambda \in P_G^+(\mu)$, $\alpha \in \Delta^+(G, T_G)$ and suppose that $\lambda - \alpha \in P_G^+(\mu)$. Then $d_\mu(\lambda - \alpha) \leq d_\mu(\lambda)$.*

Proof. It is enough to show the claim for a simple root, in which case it follows from observation

- $(\alpha_1, 0) = (\varpi_1, -\varpi_1) + (\varpi_1, \varpi_n) - (\varpi_2, \varpi_n - \varpi_1)$,
- $(\alpha_i, 0) = -(\varpi_{i-1}, -\varpi_{i-1}) + (\varpi_i, -\varpi_i) + (\varpi_i, \varpi_n - \varpi_{i-1}) - (\varpi_{i+1}, \varpi_n - \varpi_i)$ for $i = 2, \ldots, n-1$,
- $(\alpha_n, 0) = -(\varpi_{n-1}, -\varpi_{n-1}) + (\varpi_n, -\varpi_n) + (\varpi_n, \varpi_n - \varpi_{n-1})$.

Indeed, subtracting a simple root decreases r_n or s_1 by at most one. □

Remark 2.6. The structure (3) of the μ-well is also available for other multiplicity free systems, see e.g. [8, 20, 12].

Example 2.7. Let $n = 2$ and take $\mu = 4\varpi_1 - \varpi_2$. In the table in Figure 2 we have listed the five quadruples (r_1, r_2, s_1, s_2) with $\min(r_2, s_1) = 0$. The corresponding λ are the elements of the bottom $B(\mu)$, i.e. those elements in $P_G^+(\mu)$ of μ-degree zero. In Figure 2 we have drawn part of the μ-well $P_G^+(\mu)$ for this example.

3. The decomposition of the H_*-module V_μ

As in Section 2, the branching rules from H to H_* can be recovered from the classical branching rules from $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{GL}(n-1, \mathbb{C})$. We record this branching rule below, for a proof see [3, Thm.4.4]. We provide an alternative proof in Remark 3.4

r_1	r_2	s_1	s_2	λ
4	0	1	0	$5\varpi_1$
3	0	0	1	$3\varpi_1 + \varpi_2$
2	1	0	2	$2\varpi_1 + 3\varpi_2$
1	2	0	3	$\varpi_1 + 5\varpi_2$
0	3	0	4	$7\varpi_2$
$\mu = 4\varpi_1 - \varpi_2$				

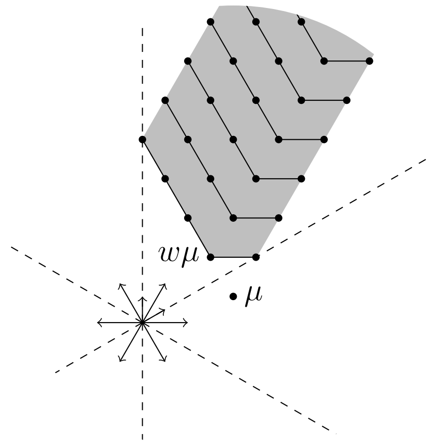

FIGURE 2. The μ-well for $(\mathrm{SL}(3,\mathbb{C}), \mathrm{GL}(2,\mathbb{C}))$ and $\mu = 4\varpi_1 - \varpi_2$.

Theorem 3.1. *Let $\mu \in P_H^+$ and $\nu \in P_{H_*}^+$ and write $\mu = \sum_{i=1}^{n+1} b_i\epsilon_i$ and $\nu = \sum_{i=1}^{n+1} c_i\epsilon_i$. Then $[\pi_\mu|_{H_*} : \pi_\nu] = 1$ if and only if* (i) *$b_i - c_i \in \mathbb{Z}$ for $2 \le i \le n$ and* (ii) *$b_i \ge c_{i+1}$ and $c_{i+1} \ge b_{i+1}$ for $1 \le i \le n-1$.*

The subgroup H_* acts spherically on H/B_H. This is a general feature for multiplicity free systems, see e.g. [20, L.2.4]. As a result the restriction $\pi_\mu|_{H_*}$ decomposes multiplicity free into irreducible H_*-representations. Note that $H_* \subset G$ is contained in the Levi subgroup with simple roots $\{\alpha_2, \ldots, \alpha_{n-1}\}$. Let $Q \subset G$ denote the parabolic subgroup with this Levi subgroup and that contains B_G. Then $Q = LU_Q$ is a Levi decomposition and the representation of L on $V_\lambda^{U_Q}$ is irreducible of highest weight λ. The group group H_* also acts irreducibly on $V_\lambda^{U_Q}$ with highest weight $\lambda_* := \lambda|_{T_{H_*}}$.

The representations of the groups induce representations Lie algebras of the groups G, H and H_*, which are all subalgebras of $\mathrm{Mat}(n \times n, \mathbb{C})$ with Lie bracket the commutator. Let $E_{i,j} \in \mathrm{Mat}(n \times n, \mathbb{C})$ be the matrix with only zeros except for a one on the j-th entry of the i-th row. It is a root vector of $\epsilon_i - \epsilon_j$ if $i \neq j$.

Proposition 3.2. *Let $\mu \in P_H^+$ and $\lambda \in P_G^+(\mu)$. Let $(r,s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ denote the pair of n-tuples that corresponds to $(\lambda, -\mu) \in \widehat{\Lambda}_+(G, B_H)$. Let $v_\mu \in V_\lambda$ be a non-zero weight vector of H of highest weight μ. Then $\prod_{i=2}^n E_{i,1}^{s_i} v_\mu$ is a non-zero weight vector for T_{H_*} of weight λ_*.*

PROOF. Since $s_{i+1} \le \mu_i$ for all $i = 1, \ldots, n-1$, the vector $\prod_{i=2}^n E_{i,1}^{s_i} v_\mu$ is non-zero. Moreover, $\prod_{i=2}^n E_{i,1}^{s_i} v_\mu$ is a weight vector of weight $\mu + \sum_{i=2}^n s_i(\epsilon_i - \epsilon_1)$. We observe that

$$\mu + \sum_{i=2}^{n} s_i(\epsilon_i - \epsilon_1) = \lambda - \sum_{i=1}^{n} s_i(\epsilon_1 - \epsilon_{n+1}). \tag{4}$$

Indeed, taking the inner product on both sides with α_j yields the coefficients of ϖ_j on both sides, which are the same because of the choice of (r,s). This settles the claim. ☐

Given $\mu \in P_H^+$ we collect the irreducible H_*-representations that occur in the decomposition of $\pi_\mu|_{H_*}$,

$$P_{H_*}^+(\mu) := \{\nu \in P_{H_*}^+ \,|\, [\pi_\mu|_{H_*} : \pi_\nu] \geq 1\}.$$

Since H/B_H is H_*-spherical, we actually have $[\pi_\mu|_{H_*} : \pi_\nu] = 1$ for all $\nu \in P_{H_*}^+(\mu)$.

Lemma 3.3. *The map $B(\mu) \to P_{H_*}^+(\mu) : \lambda \mapsto \lambda_*$ is a bijection.*

Proof. The map is injective. It is also surjective, which is a general feature of multiplicity free systems, see [20, Thm.3.1]. □

Remark 3.4. Proposition 3.2 and Lemma 3.3 show that $\nu \in P_{H_*}^+(\mu)$ if and only if it is of the form $\nu = \lambda_*$ for some $\lambda \in P_G^+(\mu)$ which in turn is equivalent to the interlacing conditions of Theorem 3.1. Hence we found a new proof of the branching rule in Theorem 3.1.

We endow the bottom $B(\mu)$ with a total ordering that is compatible with the usual partial ordering on P_G^+. Using the the identification $B(\mu) \to P_{H_*}^+$ from Lemma 3.3 we write

$$P_{H_*}^+(\mu) = \{\nu_1, \ldots, \nu_N\}, \tag{5}$$

where $N = \dim(\mathrm{End}_{H_*}(V_\mu))$ is the number of irreducible constituents of $V_\mu|_{H_*}$ and $\nu_i \leq \nu_j$ if and only if $i \leq j$.

4. Equivariant embeddings

We will study the H-equivariant embeddings $V_\mu \to V_\lambda$ by passing to bigger representation spaces, so called ambient spaces. Recall that the fundamental representations of $\mathrm{SL}(n+1, \mathbb{C})$, i.e. those whose highest weight is a fundamental weight ϖ_i, are realized as the natural representation on $\bigwedge^i \mathbb{C}^{n+1}$. An orthonormal basis $\bigwedge^i \mathbb{C}^{n+1}$ of weight vectors is given by $e_{(j_1,\ldots,j_i)} := e_{j_1} \wedge \ldots \wedge e_{j_i}$, where $1 \leq j_1 < \cdots < j_i \leq n+1$.

Definition 4.1. *For $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in P_G^+$ we define $S(G, \lambda) = \bigotimes_{i=1}^n S^{\lambda_i}(\bigwedge^i(\mathbb{C}^{n+1}))$, the ambient G-module for V_λ.*

For $\mu = \sum_{i=1}^n \mu_i \varpi_i \in P_G^+$ we define $S(H, \mu) = \bigotimes_{i=1}^{n-1} S^{\mu_i}(\bigwedge^i(\mathbb{C}^n)) \otimes \mathbb{C}_{\mu_n}$, the ambient H-module for V_μ. Here $\mathbb{C}_{\mu_n}$ is the representation space for the representation $h \mapsto \det(h)^{\mu_n}$.

Note that $\mathbb{C}_{\mu_n}$ can be identified with $S^{\mu_n}(\bigwedge^n \mathbb{C}^n)$ if $\mu_n \geq 0$ and with $S^{|\mu_n|}(\mathbb{C}e_{n+1})$ if $\mu_n \leq 0$. Here $h \in H$ acts on e_{n+1} by multiplication with $\det(h)^{-1}$. Recall that the symmetric powers of fundamental representations are in general reducible. For later reference we record the following result.

Proposition 4.2. *(a) The weight space of $S(G, \lambda)$ is weight λ is one-dimensional. (b) The weight space of $S(H, \mu)$ is weight μ is one-dimensional.*

Moreover, if $\lambda, \lambda' \in P_G^+(\mu)$ and $V_{\lambda'} \subset S(G, \lambda)$, then $d_\mu(\lambda') \leq d_\mu(\lambda)$ and $\lambda' \leq \lambda$ in the usual partial order.

Proof. We only address the statement of the degree, which follows from Lemma 2.5. □

Remark 4.3. If $n = 2$ and $\lambda, \lambda' \in P_G^+(\mu)$ with $\lambda \neq \lambda'$ and $V_{\lambda'} \subset S(G,\lambda)$, then $d_\mu(\lambda') < d_\mu(\lambda)$. Indeed, the weights of $S^{\lambda_i}(\bigwedge^i \mathbb{C}^3)$ other than $\lambda_i\varpi_i$ are of the form $\lambda_i\varpi_i - \alpha_i - \dots$. Hence $\lambda' = \lambda - n_1\alpha_1 - n_2\alpha_2$ with $n_1, n_2 \in \mathbb{N}$. But $\alpha_1 + \alpha_2 = \varpi_1 + \varpi_2$, so the degree must drop.

This is no longer the case if $n \geq 3$. Indeed, not all symmetric powers of the fundamental representations are irreducible, so we loose control over the weights.

Definition 4.4. *We fix a G-equivariant isometric embedding $S(\lambda) : V_\lambda \to S(G,\lambda)$ by fixing a highest weight vector $v_\lambda \in V_\lambda$ of length one, which we send to $e_1^{\lambda_1} \otimes \cdots \otimes e_{(1,\dots,n)}^{\lambda_n}$. Similarly we fix an H-equivariant isometric embedding $S(\mu) : V_\mu \to S(H,\mu)$ by fixing a highest weight vector $v_\mu \in V_\mu$ of length one, which we send to $e_1^{\mu_1} \otimes \ldots \otimes e_{(1,\dots,n-1)}^{\mu_{n-1}} \otimes e_{(1,\dots,n)}^{\mu_n}$ if $\mu_n \geq 0$ and to $e_1^{\mu_1} \otimes \ldots \otimes e_{(1,\dots,n-1)}^{\mu_{n-1}} \otimes e_{n+1}^{-\mu_n}$ otherwise.*

Let $S^*(\lambda) : S(G,\lambda) \to V_\lambda$ and $S^*(\mu) : S(G,\mu) \to V_\mu$ denote the equivariant orthogonal projections that satisfy $S^*(\lambda) \circ S(\lambda) = \mathrm{Id}$ and $S^*(\mu) \circ S(\mu) = \mathrm{Id}$.

We explain how to embed $S(H,\mu)$ into $S(G,\lambda)$. To this end we consider the the restriction of $S(G,\lambda)$ to H and the embedding of $S(H,\mu)$ into an even bigger tensor product. The match between these representations is given by the pair $(r,s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ that determines the pair $(\lambda^*, \mu) \in \widehat{\Lambda}_+(G, B_H)$. We start with the decomposition of the H-module $S(G,\lambda)$ into specific H-submodules.

Lemma 4.5. $S^{\lambda_i}(\bigwedge^i(\mathbb{C}^{n+1})) = \bigoplus_{u_i+v_i=\lambda_i} S^{v_i}(\bigwedge^{i-1}\mathbb{C}^n) \otimes \mathbb{C}_{-v_i} \otimes S^{u_i}(\bigwedge^i \mathbb{C}^n)$ *as H-modules.*

Proof. This follows from the decomposition (2) as H-modules together with properties of the symmetric products. □

Corollary 4.6. *As H-module we have*

$$S(G,\lambda) = \bigoplus_{(u,v)\colon u_i+v_i=\lambda_i} \left(S^{u_1}(\bigwedge^1 \mathbb{C}^n) \otimes S^{v_2}(\bigwedge^1 \mathbb{C}^n) \right) \otimes \cdots$$
$$\cdots \otimes \left(S^{u_{n-1}}(\bigwedge^{n-1} \mathbb{C}^n) \otimes S^{v_n}(\bigwedge^{n-1} \mathbb{C}^n) \right) \otimes \left(S^{u_n}(\bigwedge^{n} \mathbb{C}^n) \otimes \mathbb{C}_{-(v_1+\cdots+v_n)} \right). \tag{6}$$

Proof. The isomorphism is obtained by applying Lemma 4.5 to each factor of $S(G,\lambda)$. The trivial factor $S^{v_1}(\bigwedge^0 \mathbb{C}^n)$ is left out and the factors $\mathbb{C}_{-v_i}$ are taken together. □

Definition 4.7. *Each pair $(u,v) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ with $u_i + v_i = \lambda_i$ for all $i = 1, \dots n$ induces a canonical G-equivariant orthogonal projection onto the summand indexed by (u,v). With $j_{(u,v)}$ we denote its G-equivariant isometric section with $p_{(u,v)} \circ j_{(u,v)} = \mathrm{Id}$.*

To describe the embedding of $S(H,\mu)$ into $S(G,\lambda)$ we have to introduce some notation. Given $a,b\in\mathbb{N}_0$ and $\rho\in\mathbb{N}_0^n$ with $|\rho|=a+b$ define

$$\mathrm{Mat}((a,b),\rho)=\{(\tau^1,\tau^2)\in\mathbb{N}_0^n\times\mathbb{N}_0^n|\ (1)\,\tau_i^1+\tau_i^2=\rho_i,\ (2)\,|\tau^1|=a,\ (3)\,|\tau^2|=b\},$$

which can be viewed as the set of matrices with coefficients in $\mathbb{N}_0$ whose i-th column adds up to ρ_i and whose first and second row add up to a and b respectively. The following result is a generalization of a special case of the Clebsch-Gordan embedding theorems for $\mathrm{SL}(2,\mathbb{C})$.

Lemma 4.8. *Let $a,b\in\mathbb{N}_0$ and $p\in\mathbb{N}$. Then*

$$i_{(a,b)}:S^{a+b}(\mathbb{C}^p)\to S^a(\mathbb{C}^p)\otimes S^b(\mathbb{C}^p):e_\rho\mapsto\sum_{\tau\in\mathrm{Mat}((a,b),\rho)}\binom{a+b}{\rho}^{-1}\left(\binom{a}{\tau^1}e^{\tau^1}\otimes\binom{b}{\tau^2}e^{\tau^2}\right)$$

is an isometric $\mathrm{GL}(p,\mathbb{C})$-equivariant map.

Proof. This is a special case of [12, L.6.2]. □

Theorem 4.9. *Let $\lambda\in P_G^+(\mu)$ and let $r,s\in\mathbb{N}_0^n$ be the pair that determines $(\lambda^*,\mu)\in\widehat{\Lambda}_+(G,B_H)$. The pair (r,s) determines an H-equivariant embedding $\iota_{(\mu,\lambda)}:S(H,\mu)\to S(G,\lambda)$ such that the composition $S^*(\lambda)\circ\iota_{(\mu,\lambda)}\circ S(\mu):V_\mu\to V_\lambda$ is injective.*

Proof. First we construct the map $\iota_{(\lambda,\mu)}$. The pair (r,s) gives rise to an H-equivariant isometric embedding

$$S(H,\mu)\to\left(\bigotimes_{i=1}^{n-1}S^{r_i}(\bigwedge^i\mathbb{C}^n)\otimes S^{s_{i+1}}(\bigwedge^i\mathbb{C}^n)\right)\otimes S^{r_n}(\bigwedge^n\mathbb{C}^n)\otimes\mathbb{C}_{-(s_1+\ldots+s_n)}. \tag{7}$$

Indeed, on the first $n-1$ factors of $S(H,\mu)$ we apply Lemma 4.8. The last factor of $S(H,\mu)$ is $\mathbb{C}_{\mu_n}$ which is equal to $\mathbb{C}_{r_n}\otimes\mathbb{C}_{-(s_1+\ldots+s_n)}$ by the conditions we imposed on (r,s).

The image of (7) is the summand with $(u,v)=(r,s)$ in the decomposition of $S(G,\lambda)$ into H-submodules by Corollary 4.6. Denote the H-equivariant isometry we obtain in this way by $\iota_{(\mu,\lambda)}:S(H,\mu)\to S(G,\lambda)$.

We are left to show that the composition $S^*(\lambda)\circ\iota_{(\lambda,\mu)}\circ S(\mu)$ is injective. To this end we calculate the image of the highest weight vector of V_μ under $\iota_{(\lambda,\mu)}\circ S(\mu)$. Then we apply $\prod_{j=2}^n E_{j,1}^{s_j}$ to obtain a vector of weight $\mu-\sum_{j=1}^{s_j}(\epsilon_j-\epsilon_1)$. This vector, $\prod_{j=2}^n E_{j,1}^{s_j}((\iota_{(\lambda,\mu)}\circ S(\mu))(v_\mu))$, is non-zero by Proposition 3.2.

We have already seen that $\mu-\sum_{j=1}^{s_j}(\epsilon_j-\epsilon_1)=\lambda-\sum_{j=1}^n s_j(\epsilon_1-\epsilon_{n+1})$. Consider

$$E_{1,n+1}^{s_1+\cdots+s_{n+1}}\prod_{j=2}^n E_{j,1}^{s_j}(\iota_{(\lambda,\mu)}\circ S(\mu)(v_\mu)), \tag{8}$$

a vector of weight λ. The weight space of $S(G,\lambda)$ of weight λ is one-dimensional by Proposition 4.2. If the vector (8) is non-zero, then it is not perpendicular to $S(\lambda)(V_\lambda)$. Hence

$\iota_{(\lambda,\mu)}(S(\mu))(V_\mu))$ is not perpendicular to $S(\lambda)(V_\lambda)$. This shows that $S^*(\lambda)\circ\iota_{(\lambda,\mu)}\circ S(\mu)$ is injective, provided (8) is non-zero. To see that this condition holds, note that

$$(9)(\iota_{(\lambda,\mu)}\circ S(\mu))(v_\mu)=e_{n+1}^{s_1}e_1^{r_1}\otimes\ldots\otimes e_{(1,\ldots,i-1,n+1)}^{s_i}e_{(1,\ldots,i)}^{r_i}\otimes\ldots\otimes e_{(1,\ldots,n-1,n+1)}^{s_n}e_{(1,\ldots,n)}^{r_n}.$$

Application of $E_{1,n+1}^{s_1+\cdots+s_n}\prod_{j=2}^n E_{j,1}^{s_j}=(e_1\partial_{e_{n+1}})^{s_1+\cdots+s_n}\prod_{j=2}^n(e_j\partial_{e_1})^{s_j}$ to this vector yields

$$(10)\quad \sum_{\sigma,\sigma^2,\ldots,\sigma^n}\left(E_{1,n+1}^{\sigma_1}\prod_{j=2}^n E_{j,1}^{\sigma_1^j}\right)e_{n+1}^{s_1}e_1^{r_1}\otimes\cdots$$
$$\cdots\otimes\left(E_{1,n+1}^{\sigma_i}\prod_{j=2}^n E_{j,1}^{\sigma_i^j}\right)e_{(1,\ldots,i-1,n+1)}^{s_i}e_{(1,\ldots,i)}^{r_i}\otimes\cdots\otimes\left(E_{1,n+1}^{\sigma_n}\prod_{j=2}^n E_{j,1}^{\sigma_n^j}\right)e_{(1,\ldots,n-1,n+1)}^{s_i}e_{(1,\ldots,n)}^{r_1},$$

where the sum is taken over all tuples $\sigma=(\sigma_1,\ldots,\sigma_n)\in\mathbb{N}_0^n$ and $\sigma^j=(\sigma_1^j,\ldots,\sigma_n^j)\in\mathbb{N}_0^n, j=2,\ldots,n$ such that $|\sigma|=|s|$ and $|\sigma^j|=s_j, j=2,\ldots,n$. Many terms will be zero. In fact, the only non-zero terms are those with $\sigma=s$ and $\sigma^j=(0,\ldots,0,s_j,0,\ldots,0)$. Hence (10) equals

$$\binom{|s|}{s}\prod_{j=1}^n s_j!\prod_{j=2}^n s_j!e_1^{s_1}e_1^{r_1}\otimes\ldots\otimes e_{(2,\ldots,i,1)}^{s_i}e_{(1,\ldots,i)}^{r_i}\otimes\ldots\otimes e_{(2,\ldots,n,1)}^{s_n}e_{(1,\ldots,n)}^{r_n},$$

which is a non-zero multiple of $S(\lambda)(v_\lambda)$. □

Remark 4.10. We say that the embedding $\iota_{(\lambda,\mu)}\circ S(\mu):V_\mu\to S(G,\lambda)$ is an approximation of the embedding $V_\mu\to V_\lambda$. It means that the submodule $V_\mu\subset S(G,\lambda)$ is seen by the submodule V_λ, i.e. the composition

$$S^*(\lambda)\circ\iota_{(\lambda,\mu)}\circ S(\mu):V_\mu\to S(H,\mu)\to S(G,\lambda)\to V_\lambda$$

is injective. There may also be other irreducible G-submodules $V_{\lambda'}\subset S(G,\lambda)$ that see V_μ, but these λ' either have lower degree or the same degree, but then $\lambda'\le\lambda$ in the usual partial ordering.

5. Application to spherical functions

Definition 5.1. *Let $\mu\in P_H^+$, $\lambda\in P_G^+(\mu)$ and let $j:V_\mu\to V_\lambda$ and $p:V_\lambda\to V_\mu$ be H-equivariant maps with j isometric and with $p\circ j=\mathrm{Id}$. The function $\Phi_\lambda^\mu:G\to\mathrm{End}(V_\mu):g\mapsto p\circ\pi_\lambda(g)\circ j$ is called the spherical function of type μ associated to λ.*

The spherical functions satisfy $\Phi_\lambda^\mu(h_1gh_2)=\pi_\mu(h_1)\Phi_\lambda^\mu(g)\pi_\mu(h_2)$ for all $h_1,h_2\in H,g\in G$. We want to describe the spherical functions for a fixed element $\mu\in P_H^+$. To this end we make a number of reductions.

First of all we restrict Φ_μ^λ to the maximal compact subgroup $U=\mathrm{SU}(n+1)$ of G. Denote $K=U(n)\subset H$. The pair (U,K) is a compact symmetric pair. There exists a one dimensional torus $A_c\subset U$ such that $U=KA_cK$. In view of this decomposition and the transformation behavior of Φ^μ_λ it is enough to understand $\Phi^\mu_\lambda|_{A_c}$. In this particular example the torus A_c consists of the elements

$$a_t := \begin{pmatrix} \cos(t) & 0 & -\sin(t) \\ 0 & I_{n-1} & 0 \\ \sin(t) & 0 & \cos(t) \end{pmatrix}, \quad t \in [0, 2\pi].$$

Let $M = Z_K(A_c)$. Then the complexification of M is equal to H_*. Note that $\Phi^\mu_\lambda(a) \in \mathrm{End}_{H_*}(V_\mu)$. As we have indicated in Section 3 the representation $\pi_\mu|_{H_*}$ is multiplicity free. As a basis of V_μ we take the union of the orthonormal bases that consist of T_{H_*}-weight vectors of the H_*-isotypical constituents. With respect to this basis the matrix $\Phi^\mu_\lambda(a)$ is block-diagonal, the blocks being multiples of the identity. The multiple is given by $\langle \pi_\lambda(a)v_\nu, v_\nu\rangle$, where $v_\nu \in V_\nu \subset V_\mu \subset V_\lambda$ is the highest weight vector of weight $\nu \in P^+_{H_*}(\mu)$.

Definition 5.2. *Let $N = \dim(\mathrm{End}_{H_*}(V_\mu))$ be the number of irreducible H_*-subrepresentations of V_μ and write $V_\mu = \oplus_{i=1}^N V_{\nu_i}$. Let $v_{\nu_i} \in V_{\nu_i} \subset V_\mu$ be a highest weight vector of H_* of weight ν_i.*

Let $\{V_1, \ldots, V_N\}$ be a collection of G-modules with a fixed compatible unitary structure. Suppose that we are given a collection of $\Gamma := \{\gamma_i : V_\mu \to V_i | i = 1, \ldots, N\}$ of H-equivariant embeddings. Define the matrix valued function

$$\Psi^\mu_\Gamma : A_c \to \mathrm{Mat}(N \times N, \mathbb{C}), \quad (\Psi_\Gamma(a))_{i,j} := \langle \pi_j(a)\gamma_j(v_i), \gamma_j(v_i)\rangle.$$

The two main examples of such collections, with $d \in \mathbb{N}_0$, are given as follows.

- Let $\Upsilon_d := \{V_\mu \to V_\lambda | \lambda \in B(\mu) + d(\varpi_1 + \varpi_n)\}$, where $V_\mu \to V_\lambda$ is an isometric H-equivariant embedding. Then $\Psi^\mu_{\Upsilon_d}(a)$ is the matrix whose columns are essentially the restricted spherical functions evaluated in a.
- By $\widetilde{\Upsilon}_d$ we denote the approximated version of Υ_d, i.e. $\widetilde{\Upsilon}_d := \{S(\mu) \circ \iota_{(\lambda,\mu)} : V_\mu \to S(G, \lambda) | \lambda \in B(\mu) + d(\varpi_1 + \varpi_n)\}$.

For convenience we introduce the notation $\Psi^\mu_d = \Psi^\mu_{\Upsilon_d}$ and $\widetilde{\Psi}^\mu_d = \Psi^\mu_{\widetilde{\Upsilon}_d}$. We close this section by relating Ψ^μ_d and $\widetilde{\Psi}^\mu_d$.

Proposition 5.3. *Fix $\mu \in P^+_H$ and a total ordering on $P^+_{H_*}(\mu)$ as in Section 3. Then*

$$\widetilde{\Psi}^\mu_d(a) = \sum_{k=0}^{d} \Psi^\mu_k(a) \cdot C^\mu(d, k),$$

where $C^\mu(d, k) \in \mathrm{Mat}(N \times N, \mathbb{C})$ and $C^\mu(d, d)$ is upper triangular and invertible.

Proof. Decompose $S(G, \lambda)$ into irreducible G-modules $V_{\lambda'}$ with $\lambda' \in P^+_G(\mu)$. Then $d_\mu(\lambda') \leq d_\mu(\lambda)$. If the degrees are the same then $\lambda' \leq \lambda$ in the usual partial ordering. This implies the decomposition and the nature of $C^\mu(d, k)$ after some bookkeeping. To show that $C^\mu(d, d)$ is invertible we have to see that the diagonal entries are non-zero, which follows from Theorem 4.9. □

Remark 5.4. In case $n = 2$ we observe that $C^\mu(d,d) = \mathrm{Id}$ because symmetric powers of fundamental representations are irreducible in this case.

Proposition 5.3 shows that $\widetilde{\Psi}^\mu_d$ can be seen as an approximation of Ψ^μ_d. The function Ψ^μ_0 is of particular interest in the link of spherical functions with matrix valued orthogonal polynomials. Indeed, the matrix weight of interest is given by

$$a \mapsto (\Psi^\mu_0(a))^* D^\mu \Psi^\mu_0(a), \tag{11}$$

where $D^\mu = \mathrm{diag}(\dim V_{\nu_1}, \ldots, \dim V_{\nu_N})$, see e.g. [20, §6].

The weight $a \mapsto (\widetilde{\Psi}^\mu_0(a))^* D^\mu \widetilde{\Psi}^\mu_0(a)$ differs from (11) by conjugation with $C^\mu(0,0)$, which is upper triangular and invertible. For certain properties of the weight this is immaterial, for example for its reducibility properties or the existence of shift operators, which are invariant for conjugation with an invertible matrix.

Once the vectors $v_{\nu_1}, \ldots, v_{\nu_N} \in V_\mu$ are determined, the functions $\widetilde{\Psi}^\mu_d$ can be implemented in a computer algebra package without too much pain.

The functions Ψ^μ_d in can also be implemented, but then we have to calculate kernels of root vectors acting on high-dimensional representation spaces. We also have to implement an invariant Hermitian inner product, for which we have to invert very big matrices. Such calculations soon take too much memory.

6. Example

As an example we calculate $\widetilde{\Psi}^\mu_0$ for the case $(G,H) = (\mathrm{SL}(4,\mathbb{C}), \mathrm{GL}(3))$ and $\mu = \varpi_1 + \varpi_2 + m\varpi_3$, with $m \geq 0$. This is a step-two representation. It turns out that we can calculate Ψ^μ_0 in this case. The bottom is given by $B(\mu) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ where the λ_i are displayed in Table 1.

i	λ_i	s_1	s_2	s_3	r_1	r_2	r_3	$\dim V_{\lambda_i}$	$\dim V_{\nu_i}$
1	$\varpi_1 + \varpi_2 + m\varpi_3$	0	0	0	1	1	m	$\frac{4(m+3)(m+4)(m+6)}{3}$	2
2	$\varpi_1 + (m+2)\varpi_3$	0	0	1	1	0	$m+1$	$\frac{(m+3)(m+4)(m+6)}{2}$	1
3	$2\varpi_2 + (m+1)\varpi_3$	0	1	0	0	1	$m+1$	$(m+2)(m+5)(m+6)$	3
4	$\varpi_2 + (m+3)\varpi_3$	0	1	1	0	0	$m+2$	$\frac{(m+4)(m+6)(m+7)}{2}$	2
$\mu = \varpi_1 + \varpi_2 + m\varpi_3$, $m \geq 0$, $\dim V_\mu = 8$.									

Table 1. Some data to calculate the spherical functions

Note that $\lambda_2 = \lambda_1 + \alpha_3$, $\lambda_2 = \lambda_1 + \alpha_2 + \alpha_3$ and $\lambda_4 = \lambda_1 + \alpha_2 + 2\alpha_3$. This shows that the total ordering on $B(\mu)$ given by $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ is compatible with the usual ordering on the weight lattice. The dimensions of the representations spaces of higest weight $\lambda_i = \lambda_{i,1}\varpi_1 + \lambda_{i,2}\varpi_2 + \lambda_{i,3}\varpi_3$ are calculated using Weyl's dimension formula,

$$\dim V_{\lambda_i} = (\lambda_{i,1}+1)(\lambda_{i,2}+1)(\lambda_{i,3}+1)\frac{\lambda_{i,1}+\lambda_{i,2}+2}{2}\frac{\lambda_{i,2}+\lambda_{i,3}+2}{2}\frac{\lambda_{i,1}+\lambda_{i,2}+\lambda_{i,3}+3}{3}.$$

The H-representation V_μ decomposes into four irreducible H_*-representations of highest weights $\nu_i := \lambda_{i,*}$. The dimensions of these representation spaces are collected in Table 1 and they add up to eight, the dimension of V_μ. The highest weight vectors for H_* are displayed in Table 2. Their lengths need not be one, but all these vectors are non-zero and all are killed by $E_{3,2}$, the root vector of the only positive root of H_*. The vectors $\iota_{(\lambda_i,\mu)}(v_\mu)$ are displayed in Table 2.

i	v_{ν_i}	$\iota_{(\lambda_i,\mu)}(v_\mu)$
1	v_μ	$e_1 \otimes e_{(1,2)} \otimes e^m_{(1,2,3)}$
2	$(2E_{3,1} - E_{3,2}E_{2,1})v_\mu$	$e_1 \otimes e^{m+1}_{(1,2,3)} e_{(1,2,4)}$
3	$E_{2,1}v_\mu$	$e_{(1,2)}e_{(1,4)} \otimes e^{m+1}_{(1,2,3)}$
4	$E_{3,1}E_{2,1}v_\mu$	$e_{(1,4)} \otimes e^{m+2}_{(1,2,3)} e_{(1,2,4)}$

TABLE 2. Higest weight vectors

The root vectors $E_{i,j}$ act by $e_i\partial_{e_j}$, and since $\iota_{(\lambda_i,\mu)}$ is H-equivariant we can calculate the vectors $\iota_{(\lambda_i,\mu)}(v_{\nu_j})$ explicitly. The element a_t acts on these vectors and to obtain

$$\langle a_t\iota_{(\lambda_i,\mu)}(v_{\nu_j}), \iota_{(\lambda_i,\mu)}(v_{\nu_j})\rangle$$

we only have to compare coefficients. The calculation is a matter of careful bookkeeping. After normalizing all weight vectors to have length one we obtain

$$(12)\quad \widetilde{\Psi}^\mu_0(a_t) = c^m \begin{pmatrix} c^2 & c^2 & c^2 & c^2 \\ c & c(2(m+2)c^2 - 2(m+1)+1)/3 & c & c((2m+7)c^2 - (2m+4))/3 \\ c & c & c(2c^2-1) & c^3 \\ 1 & (m+2)c^2 - (m+1) & c^2 & c^2((m+3)c^2 - (m+2)) \end{pmatrix},$$

where $c = \cos(t)$. Now we explain how to obtain Ψ^μ_0. To this end we collect the sub-representations of the ambient representation spaces $S(G,\lambda_i)$ whose highest weights are in $P^+_G(\mu)$:

- $S(G,\lambda_1) = V_{\lambda_1}$+other summands,
- $S(G,\lambda_2) = V_{\lambda_2}$+other summands,
- $S(G,\lambda_3) = V_{\lambda_3} + V_{\lambda_1}$+other summands,
- $S(G,\lambda_4) = V_{\lambda_4} + V_{\lambda_2}$+other summands.

We will not prove this statement. The decompositions for small values of m are found using computer algebra. This yields an Ansatz for the decompositions in general. The numbers of summands are bounded uniformly which should indicate how to prove the decompositions in general. It follows that $C^\mu(0,0)$ is of the form

$$C^\mu(0,0)=\begin{pmatrix}1 & 0 & 1-c_1 & 0\\ 0 & 1 & 0 & 1-c_2\\ 0 & 0 & c_1 & 0\\ 0 & 0 & 0 & c_2\end{pmatrix}$$

where $0<c_1,c_2\leq 1$. The function $a\mapsto(\Psi_0^\mu(a))^* D^\mu\Psi_0^\mu(a)$ that we discussed below (11) is a matrix valued polynomial W_{pol}^μ in the variable $\cos^2 t$. It can be tracked back to a function on U where it can be integrated component wise against the normalized Haar measure du. By Schur orthogonality the outcome is a diagonal matrix with entries $(\dim V_\mu)^2/\dim V_{\lambda_i}$. Another way to calculate this integral is by

$$\frac{1}{3}\int_0^1 W_{\mathrm{pol}}^\mu(x)(1-x)^2dx,$$

see e.g. [19, L.3.5.7]. Putting all this information together we find $c_1=3/(m+4)$ and $c_2=3/(m+5)$, so that

$$(13)\quad \Psi_0^\mu(a_t)=$$
$$c^m\begin{pmatrix}c^2 & c^2 & c^2 & c^2\\ c & \frac{c(2(m+2)c^2-2(m+1)+1)}{3} & c & c((m+3)c^2-(m+2))\\ c & c & \frac{c(2(m+4)c^2-(2m+5))}{3} & \frac{c((m+5)c^2-(m+2))}{3}\\ 1 & (m+2)c^2-(m+1) & \frac{(m+4)c^2-(m+1)}{3} & \frac{(m+5)(m+3)c^4-(m+2)(2m+7)c^2+(m+2)(m+1)}{3}\end{pmatrix},$$

where $c=\cos(t)$. Note that $\det\Psi_0^\mu(a_t)=\frac{4(m+2)(m+3)(m+4)(m+5)}{27}\left(\cos^2(t)\right)^{4+4m}\left(1-\cos^2(t)\right)^4$.

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