WHK/SHK-PROJECT: REPRESENTATION THEORY, COMPUTER ALGEBRA AND SPECIAL FUNCTIONS

MAARTEN VAN PRUIJSSEN

Time frame: 9,5 hrs per week, for a period of 3 months.

1. Introduction

Various classes of special functions can be realized as matrix coefficients on Lie groups, and their special properties may be deduced from the representation theory. As an example, we mention the Jacobi polynomials in one or more variables, which are realized as matrix coefficients on a symmetric space G/K. It was noted by Koornwinder, that the geometric parameters (half the root multiplicities), could be varied continuously. Eric Opdam and Gert Heckman investigated these functions further in the nineteen eighties. At certain points, they used computer algebra to verify, or find, certain conjectures and formulas, which they were later able to prove rigorously. These investigations later culminated in the well known theory of Heckman-Opdam polynomials, a specific class of special functions associated to root systems.

In the nineteen nineties, a renewed interest came about in matrix valued analogues of orthogonal polynomials. Such families with properties similar to Jacobi polynomials were found by analyzing matrix coefficients associated to the compact Riemann symmetric space $\mathrm{SU}(3)/\mathrm{U}(2) \cong \mathbb{P}^2(\mathbb{C})$ [1]. Later, these examples were understood in the light of a more general construction [2, 6].

We are now at a point where we have many examples of matrix valued orthogonal polynomials available, even in higher rank, i.e. in several variables, but performing calculations with them becomes increasingly difficult. Moreover, there is evidence that these polynomials fit into a general framework, where the geometric parameters may be varied continuously [8, 9]. As in the case of the Heckman-Opdam theory, we intend to implement the basic data in GAP, with the goal of generating concrete examples on which we can test our conjectures, so that the theory can be extended.

2. Goal of the Project

Main Goal: The candidate should develop a code that produces the key function Φ_0, a matrix valued function on an r-dimensional torus, whose entries are given by specific matrix coefficients. For the special case concerning the symplectic groups, such a code is already available [8]. The function Φ_0 contains a lot of information of the families of matrix valued orthogonal polynomials, see e.g. [8, 4, 5]. In companion, the codes and routines that are being programmed should be supplied with ample documentation. In this way, other people can also benefit from the code. Once such an implementation is available, there are various possibilities.

Follow-up projects:

- Build an online-interface that produces examples of matrix valued orthogonal polynomials (see [8]).
- Investigate for which data in the classification the construction still goes through (see [6]). At this moment, Guido Pezzini and myself are trying to understand this data better, but it is helpful to have concrete examples.
- Investigate the data for which it is plausible that the parameters may be varied.

3. Description of the project

From the general theory (see [6]) the following data is given: a Lie algebra $\mathfrak{g}$, Lie subalgebras $\mathfrak{a}, \mathfrak{h} \subset \mathfrak{g}$, where $\mathfrak{a}$ is a torus of dimension r and $\mathfrak{h}$ reductive, a weight $\mu \in \mathfrak{t}_H^*$ and a finite set of weights $B(\mu) \subset \mathfrak{t}_G^*$, where $\mathfrak{t}_H, \mathfrak{t}_G$ are maximal tori in $\mathfrak{h}, \mathfrak{g}$ respectively.

General representation theory [3] now gives us irreducible representations V_μ^H and $V_\lambda^G, \lambda \in B(\mu)$, of H and G, respectively. Let H_0 and G_0 be maximal compact subgroups of G and H respectively. The main goal of the project amounts to:

(1) Find explicit bases of the spaces V_μ^H and $V_\lambda^G, \lambda \in B(\mu)$, orthonormal with respect to an inner product that is invariant for H_0 and G_0 respectively. (Implementation of Shapovalov from).

(2) Find the embeddings $\beta_{\mu,\lambda} : V_\mu^K \to V_\lambda^G, \lambda \in B(\mu)$ explicitly. (Highest weight theory).

(3) Let $A \subset G$ be the torus with Lie algebra $\mathfrak{a}$. Calculate the (entries of the) matrix valued function $A \ni a \mapsto \beta^* \circ \pi_\lambda^G(a) \circ \beta \in \mathrm{End}(V_\mu^K)$.

(4) All these calculations should be performed in GAP.

4. The Candidate

There are at least two challenges for the candidate: he or she should be(come) familiar with the basics of the representation theory of compact Lie algebras and he or she should be(come) familiar with the computer algebra package GAP. The level of the project is advanced bachelor or master.

5. Outlook

The project is a success, when some new functions Φ_0 can be produced and if all the software is well documented. If there is time left, the candidate may focus on one of the follow-up projects. Ideally, this project yields scientific output.

References

[1] F.A. Grünbaum, I. Pacharoni, J.A. Tirao, Matrix valued spherical functions associated to the complex projective plane, J. Funct. Anal. 188 (2002), 350-441.

[2] G. Heckman, M. van Pruijssen, Matrix valued orthogonal polynomials for Gelfand pairs of rank one, to appear in Tohoku.

[3] A.W. Knapp, Lie Groups Beyond an Introduction, Second Edition, Progress in Mathematics 140, Birkhäuser, Boston, 2002.

[4] E. Koelink, M. van Pruijssen, P. Román, Matrix Valued Orthogonal Polynomials related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$, Part I, Int Math Res Notices (2012) 2012 (24): 5673–5730.

[5] E. Koelink, M. van Pruijssen, P. Román, Matrix Valued Orthogonal Polynomials associated to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$, Part II, Publ. RIMS Kyoto Univ. 49 (2013), 271–312.

[6] M. van Pruijssen, Matrix valued orthogonal polynomials related to compact Gelfand pairs of rank one, thesis, Nijmegen 2012.

[7] M. van Pruijssen, Multiplicity free induced representations and orthogonal polynomials, submitted.

[8] M. van Pruijssen, P. Román, Matrix valued classical pairs related to compact Gelfand pairs of rank one, SIGMA 10 (2014), 113, 28 pages.

[9] M. van Pruijssen, P. Román, Deforming group parameters for matrix-valued orthogonal polynomials related to Gelfand pairs, in progress.

Universität Paderborn, Institut für Mathematik, Warburger Str. 100, D-33098 Paderborn
E-mail address: `vanpruijssen@math.upb.de`