MULTIPLICITY FREE INDUCED REPRESENTATIONS AND ORTHOGONAL POLYNOMIALS

MAARTEN VAN PRUIJSSEN

Dedicated to Gert Heckman for his 60th birthday.

ABSTRACT. In this paper we classify the multiplicity free systems and we give examples of families of matrix valued orthogonal polynomials that generalize Jacobi polynomials with geometric parameters. Let (G, H) be a reductive spherical pair and $P \subset H$ a parabolic subgroup such that (G, P) is a spherical pair. The triples (G, H, P) with this property are called multiplicity free systems and they are classified in this paper. We fix such a triple and let $\mu \in P_H^+$ be a character of P. We denote by $\pi_\mu^H = \mathrm{ind}_P^H(\mu)$ the Borel-Weil realization of the irreducible H-representation of highest weight μ and consider the representation $\mathrm{ind}_P^G(\mu) = \mathrm{ind}_H^G(\pi_\mu^H)$ of G, which we call a multiplicity free induced representation of G. Some properties of the spectrum of the multiplicity free induced representations are discussed. For three multiplicity free systems the spectra are calculated explicitly. The spectra give rise to families of multi-variable matrix valued orthogonal polynomials which generalize families of Jacobi polynomials: they are simultaneous eigenfunctions of a commutative algebra of differential operators, they satisfy recurrence relations and they are orthogonal with respect to integrating against a matrix weight on a compact subset.

1. INTRODUCTION

The aim of this paper is to classify multiplicity free systems and to indicate a relation, by means of examples, between multiplicity free systems and systems of vector valued orthogonal polynomials. To make this more precise, we introduce some terminology. Let G be a connected reductive group defined over $\mathbb{C}$.

Definition 1.1. *A G-variety X is called spherical if it is normal and if it admits an open orbit for the action of a Borel subgroup $B \subset G$. A pair (G, F) consisting of a reductive group G and a closed algebraic subgroup F is called spherical if G/F is a spherical G-variety.*

Definition 1.2. *A triple (G, H, P) is called a multiplicity free system if G is connected, reductive, $H \subset G$ is connected, reductive and $P \subset H$ is a parabolic subgroup such that (G, P) is spherical.*

Let X be a G-variety and let $B \subset G$ be a Borel subgroup. The characters with which B acts on the rational function field $\mathbb{C}(X)$ constitute a free subgroup $\Lambda(X)$ of $X^*(B)$, the group of all characters of B. The *rank* of the G-variety X is defined by $r_G(X) = \mathrm{rank}(\Lambda(X))$. The *complexity* $c_G(X)$ of the G-variety X is defined as the codimension of a generic B-orbit. Thus, a normal G-variety is spherical if and only if its complexity is zero.

Let (G, H, P) be a multiplicity free system. Let $\mu : P \to \mathbb{C}^\times$ be a character such that $\pi_\mu^H = \mathrm{ind}_P^H(\mu)$ is an irreducible H-representation of highest weight μ. The G-representation

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$\mathrm{ind}_H^G(\pi_\mu^H)$ decomposes into irreducible G-representations whose multiplicities are one. Indeed, the G-representations $\mathrm{ind}_H^G(\pi_\mu^H) = \mathrm{ind}_P^G(\mu)$ can be realized in the sections of a G-equivariant line bundle over G/P. Any such G-representation decomposes multiplicity free, as G/P is spherical by assumption. Such a representation is called a multiplicity free induced G-representation.

A subgroup $H \subset G$ is called *symmetric*, if there is a non-trivial involutive G-automorphism θ with $G_0^\theta \subset H \subset G^\theta$, where G^θ is the set of fixed points of θ and $G_0^\theta \subset G^\theta$ the connected component of the identity. The pair (G, H) is called symmetric, if $H \subset G$ is a symmetric subgroup. Symmetric subgroups are spherical subgroups.

Results. Our first main result is Theorem 2.1, in which we give the classification of all multiplicity free systems (G, H, P) for which G is simple, H connected, reductive and not symmetric. Since the multiplicity free systems (G, H, P) where (G, H) is a symmetric pair, have been classified in [16], the result of Theorem 2.1 completes the classification.

We are interested in these systems for the following reason. Given a multiplicity free system (G, H, P) with (G, H) symmetric, one can consider the H-biinvariant polynomials on G. This is a family of Heckman-Opdam polynomials [18] with geometric parameters. These orthogonal polynomials have the Sturm-Liouville property of being a family of simultaneous eigenfunctions of a commutative algebra of differential operators. The heuristic reason that we obtain families of special functions, is that the induction of the trivial H-representation to G, decomposes multiplicity free. However, a multiplicity free system provides many more irreducible H-representations that give multiplicity free induction to G (if P is a proper parabolic subgroup of H).

This is exploited in [19], where a construction of vector valued orthogonal polynomials for multiplicity free systems (G, H, P) where (G, H) is a spherical pair of rank one was proposed. In fact, we can associate such a family of polynomials to each character $\mu : P \to \mathbb{C}^\times$ with $\mu \in P_H^+$. All these families of polynomials have the Sturm-Liouville property. If we follow the construction for the trivial H-representation, we get back to the Heckman-Opdam polynomials, which in this case are just Jacobi polynomials.

Our second main result is the proof of existence of families of vector valued orthogonal polynomials in several variables (in Section 6), having the Sturm-Liouville property. The existence is based on an *ad hoc* analysis of the irreducible G-representation that occur in the multiplicity free induced representation, but the construction is the same as that of [19].

Related work. Our $\mathbb{C}^N$-valued orthogonal polynomials in r variables have the property that for every multi-degree $(d_1, \ldots, d_r)$, there are exactly N polynomials having highest degree $(d_1, \ldots, d_r)$. This is why the polynomials in [19] are grouped together into matrix valued orthogonal polynomials. This class of special functions forms the initial motivation of this research.

The use of representation theory in a quest for families of matrix valued orthogonal polynomials that have the Sturm-Liouville property, appears already in [15]. The idea of finding such families of functions using the representation theory of compact groups is pushed further in [28, 29] based on ideas from [30], to end in a general construction of families of matrix valued orthogonal polynomials for spherical pairs of rank one in [19].

Future work. We may even get rid of more symmetries. Indeed, in [37], using the construction in [19], all $\mathbb{C}^2$-valued polynomials in one variable have been calculated. It turns out that the root multiplicities, that come from the symmetric space (underlying) G/H, may be varied continuously. However, it is only speculation that there exists a theory of vector valued orthogonal polynomials associated to root systems, that contains our class of examples as a special case.

Other methods to attack problems concerning multiplicity free representations are the theories of visible actions and propagation of multiplicity free representations of Kobayashi [26, 27]. It would be interesting to see how our multiplicity freeness results fit into this picture.

Another point that is not addressed in this paper, is the study of spherical invariants of the subgroups $P \subset G$. We have used classical methods instead, but it would be interesting to see how to obtain our classification using spherical data, see e.g. [5]. Such an approach could also give an effective method to calculate which irreducible G-representations occur in $\mathrm{ind}_P^G(\mu)$. For example, for the spherical pairs (G,S) where $S \subset G$ is solvable, the spectra of $\mathrm{ind}_S^G(\chi)$ are calculated in [3], by means of the extended weight semi-group. The problem that is solved in [3], is of the same nature, namely the classification of spherical subgroups of a given kind, that are contained in a spherical subgroup of a given kind. The example $(\mathrm{SL}_{n+1}, \mathrm{GL}_n, B_{\mathrm{GL}_n})$, where $B_{\mathrm{GL}_n} \subset \mathrm{GL}_n$ is a Borel subgroup, has both flavors.

1.1. **Notation and conventions.** The following notations and conventions are employed in this paper. Groups are indicated with Latin capitals, their Lie algebras with their gothic counterparts. The character group of G is denoted by $X^*(G)$. The roots and weights that occur are numbered as in [22, App. C]. After we have fixed a Borel subgroup of a reductive group G, its weight monoid is denoted by P_G^+. Given a weight $\lambda \in P_G^+$, an irreducible representation of G of highest weight λ is denoted by π_λ^G and its representation space by V_λ^G. The restriction $\pi_\lambda^G|_H$ of the irreducible representation π_λ^G to a reductive subgroup $H \subset G$ decomposes into irreducible H-representations π_μ^H. Their multiplicities are denoted by $m_\lambda^{G,H}(\mu)$. The dual vector space of a vector space V is denoted by V^*. The highest weight of the irreducible representation contragredient to π_λ^G is denoted by λ^*.

Let $G : X$ be a reductive group acting on a quasi-affine variety and fix $\chi \in X^*(G)$. By $\mathbb{C}[X]_{(\chi)}^{(G)}$ we denote the set of elements $f \in \mathbb{C}[X]$ such that $(g \cdot f)(x) := f(g^{-1} \cdot x) = \chi(g) f(x)$ for all $x \in X, g \in G$.

The embeddings of the subgroups $H \subset G$ of items №1–12 are given in [31], those of №13–20 are given in [7].

2. Classification I

The notion of a multiplicity free system depends only on the Lie algebras, so for the classification it is sufficient to look at those triples (G,H,P) with G simply connected. Moreover, we classify only the indecomposable triples, i.e. that are not of the form $(G_1 \times G_2, H_1 \times H_2, P_1 \times P_2)$ where (G_i, H_i, P_i) are multiplicity free systems. Moreover, we assume all subgrous H to be connected. The group G may be assumed to be semi-simple because a

№	G	H	J_H^c
1a	SL_{n+m}	$\mathrm{SL}_m \times \mathrm{SL}_n$ $m > n \geq 3$	$\{\alpha'_1\}, \{\alpha'_{n-1}\}$
1b	SL_5 SL_6 $\mathrm{SL}_{n+2}\, (n \geq 5)$	$\mathrm{SL}_3 \times \mathrm{SL}_2$ $\mathrm{SL}_4 \times \mathrm{SL}_2$ $\mathrm{SL}_n \times \mathrm{SL}_2$	$\{\alpha'\}$ $\{\alpha_1\}, \{\alpha_3\}, \{\alpha'\}$ $\{\alpha_j\}, \{\alpha'\}, j = 1, \ldots, n-1$
1c	$\mathrm{SL}_{m+1}\, (m \geq 2)$	SL_m	$\Pi_H \backslash \{\alpha_i\}$
2	$\mathrm{SL}_{2n+1}\, (n > 1)$	$\mathrm{Sp}_{2n} \times \mathbb{C}^\times$	$\emptyset$
3	SL_{2n+1}	Sp_{2n}	$\emptyset$
4	Sp_{2n}	$\mathrm{Sp}_{2n-2} \times \mathbb{C}^\times$	$\alpha_i \in J^c \Rightarrow \alpha_{i\pm1} \notin J_H^c$
5	SO_{2n+1}	GL_n	$\emptyset$
6	SO_{4n+2}	SL_{2n+1}	$\{\alpha_1, \alpha_{2n}\}$
7	SO_{10}	$\mathrm{Spin}_7 \times \mathrm{SO}_2$	$\emptyset$
8	SO_9	Spin_7	$\{\alpha_1\}$
9	SO_8	G_2	$\emptyset$
10	SO_7	G_2	$\{\alpha_1\}, \{\alpha_2\}$
11	E_6	Spin_{10}	$\emptyset$
12	G_2	SL_3	$\{\alpha_1\}, \{\alpha_2\}$
13	$\mathrm{SL}_n \times \mathrm{SL}_{n+1}$	$\mathrm{SL}_n \times \mathbb{C}^\times$	$\emptyset$
14	$\mathrm{Sp}_{2n} \times \mathrm{Sp}_4$	$\mathrm{Sp}_{2n-4} \times \mathrm{Sp}_4$	$\emptyset$
15	$\mathrm{SL}_n \times \mathrm{Sp}_{2m}$ $n \geq 2$	$\mathrm{GL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$	$\{\alpha'\}, \{\alpha''_j\}$ $j = 1 \ldots, m-1$
16	$\mathrm{SO}_n \times \mathrm{SO}_{n+1}$	SO_n	$\emptyset$
17	$\mathrm{SL}_n \times \mathrm{Sp}_{2m}$ $n \geq 5$	$\mathrm{GL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$	$\{\alpha'\}, \{\alpha''_j\}$ $j = 1 \ldots, m-1$
18	$\mathrm{Sp}_{2l} \times \mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n}$	$\mathrm{Sp}_{2l-2} \times \mathrm{Sp}_{2m-2} \times \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2$	$\emptyset$
19	$\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n}$	$\mathrm{Sp}_{2m-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$	$\{\alpha_i\}, \{\alpha'\}, \{\alpha''_j\}$ $i = 1 \ldots, n-1$ $j = 1 \ldots, m-1$
20	$\mathrm{Sp}_{2m} \times \mathrm{Sp}_4 \times \mathrm{Sp}_{2n}$	$\mathrm{Sp}_{2m} \times \mathrm{Sp}_2 \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2n}$	$\emptyset$

Table 1. Multiplicity free systems (G, H, P) where (G, H) spherical, non-symmetric and G simple. The standard parabolic subgroups are parametrized by subsets $J_H \subset \Pi_H$. We have indicated complements J_H^c of the possible subsets of Π_H. If H has multiple factors, then zero, one or two primes indicate to which factor the root belongs.

possible center is always contained in any Borel subgroup. If (G, H, P) is a multiplicity free system, then (G, H) is a spherical pair and these have been classified by Krämer [31], Brion [7] and Mikityuk [34]. As a result, the list of candidates is short. Moreover, we need not be concerned with the spherical pairs that are symmetric, as those multiplicity free systems have been classified in [16].

Theorem 2.1. *The multiplicity free systems (G, H, P) with G simply connected, $H \subset G$ reductive and connected and (G, H) not symmetric, are classified in Table 1.*

The proof of this result is a case-by-case analysis and is carried out in the rest of this section and the final part of Section 4. Let $H \subset G$ be a reductive spherical subgroup. If $P \subset H$ is spherical in G, then $\dim P + \dim B_G \geq \dim G$. This criterion takes care of four items in Table 1.

Lemma 2.2. *The subgroups H of G in items 7,9,13 and 16 admit no proper parabolic subgroups $P \subset H$ for which (G, H, P) is a multiplicity free system.*

Proof. Let $g = \dim G, b_G = \dim B_G$ and $h = \dim H$. If $g = h + b_G$, then there is no need to calculate the dimensions of the maximal proper parabolic subgroups of H. The result follows from the various dimensions, which we collected in Table 2. □

№	g	b_G	h	p_M
7	45	25	22	15,16,17
9	28	16	14	9
13	$2n^2+2n-1$	n^2+2n-1	n^2	–
16a	$4n^2$	$2n^2+n$	$2n^2-n$	–
16b	$4n^2+4n+1$	$2n^2+3n+1$	$2n^2+n$	–

Table 2. Dimensions of the groups G, B_G, H and of the maximal proper parabolic subgroups of H, in case $g - b_G - h > 0$, indicated in the column p_M.

The following result has been established in e.g. [25, Lemma 5.2] and [35, §2.1].

Lemma 2.3. *Let $B \subset G$ be a Borel subgroup such that $BH \subset G$ is open. Then $B \cap H$ is a Borel subgroup of a general isotropy group $H_* \subset H$ for H acting on $\mathfrak{h}^\perp \subset \mathfrak{g}^*$.*

We use this result to provide a criterion for a triple (G, H, P) to be a multiplicity free system.

Proposition 2.4. *Let G be a reductive group and $H \subset G$ a reductive spherical subgroup. Let $P \subset H$ be a parabolic subgroup and let H_* be a general isotropy group for H acting on $\mathfrak{h}^\perp$. The pair (G, P) is spherical if and only if H/P is H_*-spherical.*

Proof. Let $B \subset G$ be a Borel subgroup such that $BH \subset G$ is open. Consider the map $c : G/P \to G/H$. Since $c^{-1}(BH/H)$ is open and B-stable, G/P has an open B-orbit if and only if $c^{-1}(BH/H) = BH/P$ has an open B-orbit. Upon writing $BH/P \cong B \times^{B\cap H} H/P$, it follows that $c^{-1}(BH/H)$ has an open B-orbit if and only if $B \cap H$ has an open orbit in the fiber $c^{-1}(H/H) = H/P$. This is equivalent to H/P being H_*-spherical. □

The generic stabilizers for actions of semisimple groups acting on vector spaces, have been calculated by Élashvili [12, 13]. For the other cases we use [25, §5].

Corollary 2.5. *The subgroups H in item 2, 3 and 5 of Table 1 do not contain proper parabolic subgroups that remain spherical in G.*

Proof. Given a proper parabolic subgroup $P \subset H$, the quotient H/P does not admit an open H_*-orbit if $\dim H/P > \dim H_*$. This proves the claim for items 3 and 5, as in these cases $\dim(H_*) = 0$.

Let $P \subset H = \mathrm{Sp}_{2n} \times \mathbb{C}^\times$ be a maximal proper parabolic subgroup. Then $\dim(H/P)$ is either $j(4n-2j+1)$ (with $1 \leq j < n$) or $\frac{1}{2}(n^2+n)$. Both cases yield $\dim H/P > 1$ (recall that $n > 1$ for this item), proving the claim for item 2, as $\dim(H_*) = 1$. □

Lemma 2.6. *The standard parabolic subgroups of H of № 8, 10 and 12 indicated in Table 1, are the only standard parabolic subgroups that remain spherical in G.*

Proof. № 10 and 12 are discussed in [19]. This leaves № 8, for which we use Proposition 2.4. There is one candidate for a parabolic subgroup P, it is determined by $\{\alpha_1\}^c$ and has dimension 16. An irreducible representation of Spin_7 of highest weight $k\omega_1$ restricted to $H_* = \mathrm{SL}_3$ decomposes multiplicity free. To see this, note that $H_* = \mathrm{SL}_3 \subset \mathrm{G}_2 \subset \mathrm{Spin}_7 = H$. The restriction $\pi^H_{k\omega_1}|_{\mathrm{G}_2}$ remains irreducible (see [19, §4]) and hence $\pi^H_{k\omega_1}|_{H_*}$ decomposes multiplicity free, according to the branching rules of G_2 to SL_3, see e.g. [17]. □

The subgroups $H \subset G$ in items 1, 6 and 11 are contained in a Levi subgroup of G, i.e. there is a parabolic subgroup $Q \subset G$ with Levi-decomposition $Q = L_Q \cdot Q_u$ and $H \subset L_Q$. In fact, $L_Q = H \cdot \mathbb{C}^\times$. Let $P \subset H$ be a parabolic subgroup with Levi-decomposition $P = L_P \cdot P_u$. The group $R = L_R \cdot R_u \subset G$ with $L_R = \mathbb{C}^\times \cdot L_P$ and $R_u = P_u \cdot Q_u$ is a parabolic subgroup and $P = L_P \cdot P_u \subset L_R \cdot R_u$ is a regular embedding, i.e. $L_P \subset L_R$ and $P_u \subset R_u$. The following Proposition is a result of Brion (see [7, Prop. 1.1]), adapted to this situation.

Proposition 2.7. *In the situation we just described, the following are equivalent:*

- *$P \subset G$ is spherical*
- *$L_P \subset L_R$ is spherical and the action of L_P on Q_u is spherical.*

Proof. Following [7, Prop. 1.1], $P \subset G$ is spherical if and only if $L_P \subset L_R$ is spherical and a Borel subgroup $B_{L_R} \subset L_R$ with $B_{L_R} \cdot L_P \subset L_R$ open, has an open orbit in R_u/P_u. The latter is L_P-equivariantly isomorphic to Q_u. Moreover, $L_P \subset L_R$ is spherical, and any Borel subgroup of L_R is of the form $\mathbb{C} \cdot B_{L_P}$, where $B_{L_P} \subset L_P$ is a Borel subgroup. □

Finally, we note that the action of L_P on Q_u is spherical if and only if the adjoint action of L_P on $\mathrm{Lie}(Q_u)$ is spherical. The representations of H on $\mathrm{Lie}(Q_u)$ are recorded in Table 3. These representations are spherical, meaning that a Borel subgroup of H has an open orbit in $\mathrm{Lie}(Q_u)$. Thus, the problem is reduced to finding the Levi subgroups $L_P \subset H$ that remain acting spherically on $\mathrm{Lie}(Q_u)$.

Spherical actions of reductive groups on vector spaces have been classified in [4, 6, 20, 33]. To check sphericity of the representation $L_P : \mathrm{Lie}(Q_u)$, we invoke the tables from [23]. We also need the following result.

The action $L_P : \mathrm{Lie}(Q_u)$ is in general not *saturated*, i.e. the dimension of the center of L_Q is less than the number of irreducible summands in the decomposition of $\mathrm{Lie}(Q_u)$ in L_P-types. We simply add $\mathbb{C}^\times$-factors to obtain a saturated representation. Suppose that the new group, denoted by L_P^{sat}, acts spherically on $\mathrm{Lie}(Q_u)$. The weights of the corresponding action of L_P^{sat} on $\mathbb{C}[\mathrm{Lie}(Q_u)]$ generate a subspace $\mathfrak{a}^*$ in the dual of a Cartan Lie algebra of L_P^{sat}. Let $\mathfrak{z}$ denote the Lie algebra of the center of L_Q^{sat} and let $\mathfrak{z}_0$ denote the Lie algebra of the center of L_P. We will use the following criterion [23, Thm. 5.1], originally due to Leahy.

Lemma 2.8. *Suppose that L_P^{sat} acts spherically on* $\mathrm{Lie}(Q_u)$. *Then L_P acts spherically on* $\mathrm{Lie}(Q_u)$ *if and only if the projection $\mathfrak{a}^* \cap \mathfrak{z}^* \to \mathfrak{z}_0^*$ is injective.*

Lemma 2.9. *Let $P = L_P P_u \subset H$ be a Levi decomposition of a parabolic subgroup of H, where $H \subset G$ is one of the items 1, 6 or 11 in Table 1. Then $P \subset G$ is spherical if and only if it is conjugated to a standard parabolic subgroup described in Table 1.*

Proof. For the items 1, 6 and 11, the Levi subgroups of $\mathbb{C} \cdot H$ that remain acting spherically on $\mathrm{Lie}(Q_u)$ have been determined in [16]. Any parabolic subgroup $P \subset H$ that is spherical in G, gives rise to a parabolic subgroup $\mathbb{C}^\times \cdot P \subset \mathbb{C}^\times \cdot H$ which is spherical in G. Thus, [16] gives candidates for Levi subgroups of H in these cases. We have recorded this information in Table 3.

№	H	$\mathrm{Lie}(Q_u)$	candidates for J_H^c
1a	$\mathrm{SL}_m \times \mathrm{SL}_n$ $m > n \geq 3$	$\mathbb{C}^m \otimes \mathbb{C}^n$	$\{\alpha_i\}$, $i = 1, m-1$, $\{\alpha'_j\}$, $j = 1, \ldots, n-1$
1b	$\mathrm{SL}_n \times \mathrm{SL}_2$ $n \geq 3$	$\mathbb{C}^n \otimes \mathbb{C}^2$	$\{\alpha_i\}, \{\alpha'\}$, $i = 1, \ldots, n-1$
1c	SL_n $n \geq 2$	$\mathbb{C}^n$	any subset of Π_H
6	SL_{2n+1}	$\bigwedge^2 \mathbb{C}^{2n+1}$	$\{\alpha_1\}, \{\alpha_{2n}\}$
11	Spin_{10}	$\mathbb{C}^{16}$	$\{\alpha_1\}$

TABLE 3. The representations of H on $\mathrm{Lie}(Q_u)$.

The notation concerning weights in this proof is as follows. The highest weight of a GL_n representation on $\bigwedge^k \mathbb{C}^n$ is denoted by ω_k. Weights of a product of groups are decorated by primes, to indicate for which factor they are a weight.

(№ 1a). Upon taking $\{\alpha_1\}$ or $\{\alpha_{m-1}\}$, we have $L_P = \mathrm{GL}_{m-1} \times \mathrm{SL}_n$ and $L_P^{sat} = \mathrm{GL}_{m-1} \times \mathrm{GL}_n$ acting on $(\mathbb{C}^{m-1} \otimes \mathbb{C}^n) \oplus \mathbb{C}^n$. If $n = m-1$ then $\mathfrak{a}^* \cap \mathfrak{z}^*$ is two dimensional. If $m-1 > n$, then $\mathfrak{a}^* \cap \mathfrak{z}^*$ is spanned by ω'_n, which vanishes on $\mathfrak{z}_0$. In both cases L_P does not act spherically.

If we take $\{\alpha_{m+1}\}$ or $\{\alpha_{m+n-1}\}$ then $L_P = \mathrm{SL}_m \times \mathrm{GL}_{n-1}$ and $L_P^{sat} = \mathrm{GL}_m \times \mathrm{GL}_{n-1}$ acts on $(\mathbb{C}^m \otimes \mathbb{C}^{n-1}) \oplus \mathbb{C}^m$. In this case, $\mathfrak{z}^* \cap \mathfrak{a}^*$ is spanned by ω'_{n-1}, which does not vanish on $\mathfrak{z}_0$. Hence, in these cases, L_P acts spherically on $\mathrm{Lie}(Q_u)$.

(№ 1b). If we take $\{\alpha_{n+1}\}$ then $L_P = \mathrm{SL}_n \times \mathbb{C}^\times$ acts on $\mathbb{C}^n \oplus \mathbb{C}^n$ via $(A, z)(v, w) = (Azv, Az^{-1}v)$ and this action is spherical.

If we take $\{\alpha_j\}$ with $j = 1, \ldots, n-1$, then $L_P = \mathrm{S}(\mathrm{GL}_j \times \mathrm{GL}_{n-j}) \times \mathrm{SL}_2$ and $L_P^{sat} \cong \mathrm{GL}_j \times \mathrm{SL}_2 \times \mathrm{GL}_{n-j}$ acts on $(\mathbb{C}^j \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{n-j})$. If $j = 1$ or $n - j = 1$, then $L_P^{sat} \cong \mathrm{GL}_2 \times \mathrm{GL}_{n-1}$ acts on $\mathbb{C}^2 \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{n-1})$. If moreover $n = 3$, then the dimension of $\mathfrak{a}^* \cap \mathfrak{z}^*$ is two, so L_P does not act spherically as $\dim \mathfrak{z}_0 = 1$. If $n > 3$, then $\mathfrak{a}^* \cap \mathfrak{z}^*$ is spanned by ω_2, which does not vanish on $\mathfrak{z}_0$. Hence, L_P acts spherically in this case.

If $j = 2 = n - 2$ then $\mathfrak{a}^* \cap \mathfrak{z}^*$ is two dimensional. In this case, L_P does not act spherically.

If $n \geq 5$ and $2 \leq j, n-j$, then $\mathfrak{a}^* \cap \mathfrak{z}^*$ is either zero-dimensional or it is spanned by the ω_2 of the factor GL_2. But ω_2 does not vanish on $\mathfrak{z}_0$. Hence L_P acts spherically.

(№ 1c). The smallest Levi subgroup of SL_n is the torus, which does not act spherically on $\mathbb{C}^n$. If we take a minimal parabolic subgroup that is not solvable, then $L_P \cong S(\mathrm{GL}_2 \times (\mathbb{C}^\times)^{n-2})$, which acts spherically on $\mathbb{C}^n$.

(№ 6). The two possible choices yield $L_P = \mathrm{GL}_{2n}$ and $L_P^{sat} = \mathbb{C}^\times \times \mathrm{GL}_{2n}$ acting on $\bigwedge^2 \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. In this case, $\mathfrak{a}^* \cap \mathfrak{z}^*$ is spanned by ω_{2n}, which does not vanish on $\mathfrak{z}_0$. Hence, L_P acts spherically.

(№ 11). We have $L_P = \mathbb{C}^\times \times \mathrm{Spin}_8$ and $L_P^{sat} = L_P \times \mathbb{C}^\times$ acts on $\mathbb{C}^8 \oplus \mathbb{C}^8$ (sum of the two Spin-representations). In this case, $\mathfrak{a}^* \cap \mathfrak{z}^*$ is two dimensional, whereas $\mathfrak{z}_0$ is of dimension one. Hence L_P does not act spherically. □

Lemma 2.10. *Concerning № 4, a standard parabolic subgroup $P \subset H$ remains spherical in G if and only if its Levi complement is equal to $\mathrm{GL}_{n_1} \times \dots \mathrm{GL}_{n_{k-1}} \times \mathrm{Sp}_{2n_k} \times \mathbb{C}^\times$, with $n_1 + \ldots + n_k = n-1, n_i \geq 0$ and $n_j \geq 2$ for $j \geq 2$.*

Proof. $H \subset G$ is a Levi subgroup itself and Proposition 2.7 applies here, too. Indeed, $L_Q = H$ and $L_P P_u \subset L_P P_u Q_u$ is regular. Hence $P \subset G$ is spherical if and only if L_P acts spherically on $\mathrm{Lie}(Q_u)$.

The group $H = \mathrm{Sp}_{2n-2} \times \mathbb{C}^\times$ acts spherically on $\mathbb{C}^{2n-2} \oplus \mathbb{C}$. The question is whether a Levi subgroup of Sp_{2n-2} remains acting spherically on $\mathbb{C}^{2n-2}$.

A Levi subgroup is typically of the form $\mathrm{GL}_{n_1} \times \dots \mathrm{GL}_{n_{k-1}} \times \mathrm{Sp}_{2n_k}$ and it acts on $\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_1 *} \oplus \cdots \oplus \mathbb{C}^{n_{k-1}} \oplus \mathbb{C}^{n_{k-1}*} \oplus \mathbb{C}^{2n_k}$. This action is spherical if and only if for all $j < k$, we have $n_j \geq 2$. This means that any subset $J_H \subset \Pi_H$ yields a Levi subgroup that remains acting spherically, provided the roots in J_H^c are not adjacent. □

Lemma 2.11. *Concerning № 14, H does not admit parabolic subgroups that remain spherical in G.*

Proof. Consider a standard parabolic subgroup $P \subset \mathrm{Sp}_{2n-4} \times \mathrm{Sp}_4$ that remains spherical in G. The group H is embedded in G via the intermediate subgroup $G_1 = \mathrm{Sp}_{2n-4} \times \mathrm{Sp}_4 \times \mathrm{Sp}_4$ via $H \to G_1, (x,y) \mapsto (x,y,y)$ and $G_1 \to G$ is the embedding $(x,y,z) \mapsto (x \oplus y, z)$. The projection of G and its subgroups H and P onto the first factor, gives the multiplicity free system $(\mathrm{Sp}_{2n}, \mathrm{Sp}_{2n-4} \times \mathrm{Sp}_4, P_1 \times P_2)$. Similarly, the projection of the multiplicity free system $(G_1, H, P_1 \times P_2)$ onto the last two factors of G_1, yields the multiplicity free systems $(\mathrm{Sp}_4 \times \mathrm{Sp}_4, \mathrm{Sp}_4, P_2)$. Following [16], we conclude that $P_2 = \mathrm{Sp}_4$ and $P_1 = \mathrm{Sp}_{2n-4}$. □

We postpone the rest of the proof of Theorem 2.1 to Section 4. We first have to develop some results concerning branching rules.

3. The spectrum

Let G be simply connected. Let $H \subset G$ be a spherical subgroup and let π_μ^H be an irreducible H-represenatation of highest weight $\mu \in P_H^+$. We are interested in the spectrum

$$P_G^+(\mu) = \{\lambda \in P_G^+ : m_\lambda^{G,H}(\mu) = 1\},$$

which we will call the μ-well. If $\mu = 0$, then $P_G^+(0)$ consists of the spherical weights. The 0-wells $P_G^+(0)$ are known for all spherical pairs (G,H), see [31] and [2] (for the cases where G is simple and not-simple, respectively). In many cases, $P_G^+(0)$ is a finitely generated, free monoid. This holds true in particular for the symmetric pairs, where $P_G^+(0)$ is understood by the Cartan-Helgason theorem (see e.g. [22, Thm. 8.49]).

The μ-wells for multiplicity free induction for multiplicity free systems (G,H,P) with (G,H) spherical of rank one, are calculated in [10, 19, 36]. In Figure 1 we have depicted the μ-well for the spherical pair $(G,H) = (\mathrm{SL}_3, \mathrm{GL}_2)$ and $\mu = 4\omega_1 - \omega_2$. In this case, the rank of

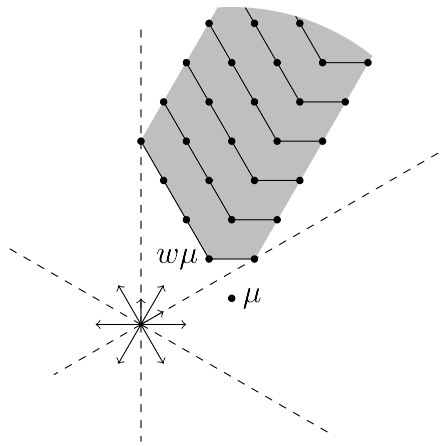


FIGURE 1. The μ-well for $(\mathrm{SL}_3, \mathrm{GL}_2)$ and $\mu = 4\omega_1 - \omega_2$.

the spherical space is one and $P_G^+(0) = \mathbb{N} \cdot \alpha_3$. From the picture we observe, that $\lambda \in P_G^+(\mu)$ implies that $\lambda + n\alpha_3 \in P_G^+(\mu)$. More generally,

$$m_{\lambda+k\sigma}^{G,H}(\mu) \leq m_{\lambda+(k+1)\sigma}^{G,H}(\mu), \tag{1}$$

for all $k \in \mathbb{N}, \mu \in P_H^+, \lambda \in P_G^+$ and $\sigma \in P_G^+(0)$, see e.g. [17, Lem. 7.1]. On the other hand, the quantity $m_\lambda^{G,H}(\mu)$ is bounded from above by a multiplicity concerning the restriction of π_μ^H to the general stabilizer H_*.

Let $B \subset G$ be a Borel subgroup such that $BH \subset G$ is open. The subgroup $Q = \{g \in G : gBH = BH\}$ contains B and is called the parabolic subgroup of G adapted to G/H. The local structure theorem for spherical varieties (see e.g. [8]) implies that Q admits a Levi decomposition $Q = Q_u L$ and L admits a subgroup L_0 with $(L, L) \subset L_0$, such that the natural map $Q \times^{L_0} L \to QH/H = BH/H$ is a Q-equivariant isomorphism. Moreover, $L_0 \cong H_*$, the general isotropy group for H acting on $\mathfrak{h}^\perp \subset \mathfrak{g}^*$, see e.g. [38, Thm. 9.1].

Let $\lambda \in P_G^+$ and let $(\pi_\lambda^G, V_\lambda^G)$ be a G-representation of highest weight λ. The Levi subgroup L acts irreducibly on $V = (V_\lambda^G)^{Q_u}$ and thus so does H_*, say with highest weight $\lambda_* \in P_{H_*}^+$. As $HB \subset G$ is dense, and B leaves V invariant, a highest weight vector in V is H-cyclic. It follows that every H-type that occurs in the restriction $\pi_\lambda^G|_H$ contains an H_*-type of highest weight λ_*. In other words, $m_\mu^{H,H_*}(\lambda_*) \geq m_\lambda^{G,H}(\mu)$ for all $\mu \in P_H^+$ (this argument was taken from [9, Rem. 1]). Define

$$P_{H_*}^+(\mu) = \{\nu \in P_{H^*}^+ : m_\mu^{H,H_*}(\nu) \geq 1\}.$$

We obtain a map $P_G^+(\mu) \to P_{H_*}^+(\mu) : \lambda \mapsto \lambda_*$. The next result by Kitagawa [21] implies that this map is surjective.

Theorem 3.1. *Let $\lambda \in P_G^+$ and let $\sigma \in P_G^+(0)$ be such that there exists a non-zero element $f \in \mathbb{C}[G]_{(\sigma,0)}^{(B\times H)}$, that cuts out the complement of HB in G. Then $m_\mu^{H,H_*}(\lambda_*) = m_{\lambda+m\sigma}^{G,H}(\mu)$ for $m \gg 0$.*

In the case where (G, H) is a symmetric pair, this result was proved by Wallach [39, Cor. 8.5.15]. Inspired by this proof, we found an argument using methods from algebraic geometry.

Proof of Theorem 3.1. Existence of σ is guaranteed by the following observation: $HB \subset G$ is affine, being the pre-image of the affine space $BH/H = B/(B \cap H)$ of the affine map $G \to G/H$, hence $BH \subset G$ is of codimension one and thus cut out by a regular function s', unique up to multiplication by an invertible element in $\mathbb{C}[G]$. As the divisor of s' is $B \times H$-stable, the function $(b,h) \cdot s'$ cuts out the same divisor. It follows that there exists $a : (B \times H) \times BH \to \mathbb{C}^\times$ with $((b,h) \cdot s')(x) = a_{(b,h)}(x)s'(x)$. This function is algebraic, and of the form $a_{(b,h)}(x) = u(b,h)v(x)$, where $u \in \mathcal{O}(B \times H)^\times$ and $v \in \mathcal{O}(BH)^\times$ (see e.g. [24]). Moreover, the function $y \mapsto a_y$ satisfies a cocycle relation $a_{y'y}(x) = a_y(y'^{-1}x)a_{y'}(x)$. Combining these results yield that v is constant and $y \mapsto a_y$ is a character.

Hence $s' \in \mathbb{C}[G]^{(B\times H)}_{(\sigma',\chi')}$ for some characters σ', χ' of B and H respectively. Let $s'' \in \mathbb{C}[G]^{(B\times H)}_{(\sigma'^*,\chi'^{-1})}$ be non-zero. Then $s's'' \in \mathbb{C}[G]^{(B\times H)}_{(\sigma,0)}$ cuts out the complement of BH in G.

Let $\mathcal{L} = \mathcal{L}_{-\lambda^*}$ be the line bundle over G/B associated to $\lambda \in P_G^+$, i.e. $\Gamma(G/B, \mathcal{L}) \cong V_\lambda^G$ as G-modules. Replace σ by a large multiple so that $\lambda - \sigma \notin P_G^+$. Let $R \subset G$ be the stabilizer of the highest weight line in V_σ^G. We realize V_σ^G as the space of sections of the line bundle $\mathcal{L}_{-\sigma^*}$ over G/R. Let $p : G/B \to G/R$ be the natural projection. Define

$$A = \bigoplus_{n\in\mathbb{N}} \Gamma(G/R, \mathcal{L}^n_{-\sigma^*}), \quad M = \bigoplus_{n\in\mathbb{N}} \Gamma(G/B, \mathcal{L} \otimes p^*\mathcal{L}^n_{-\sigma^*}).$$

In view of the isomorphism $\Gamma(G/B, p^*\mathcal{L}_{-\sigma^*}) = \Gamma(G/R, \mathcal{L}_{-\sigma^*})$ as G-modules, the graded A-module M defines a coherent sheaf $\widetilde{M}$ over G/R. Moreover,

$$\Gamma_*(p_*\mathcal{L}) = \bigoplus_{n\in\mathbb{N}} \Gamma(G/R, p_*\mathcal{L} \otimes \mathcal{L}^n_{-\sigma^*}) = M$$

by the product formula, whence $p_*\mathcal{L} = \widetilde{M}$. Let $v_H \in V_\sigma^G$ be a non-trivial H-fixed vector and let $D_+(v_H) \subset G/R$ denote the complement of the H-invariant divisor (v_H) on G/R. According to the previous observations there is an H-equivariant isomorphism $\Gamma(D_+(v_H), \widetilde{M}) \cong \Gamma(HB/B, \mathcal{L})$ because $p^{-1}(D_+(v_H)) = HB/B \cong H/B_*$. Induction in stages implies

$$\Gamma(HB/B, \mathcal{L}) = \operatorname{ind}_{B_*}^H(-\lambda^*)|_{B_*} = \operatorname{ind}_{H_*}^H \pi_{\lambda_*}^{H_*}.$$

On the other hand, the space $\Gamma(D_+(v_H), \widetilde{M})$ is the direct limit of the system $V^G_{\lambda+k\sigma} \to V^G_{\lambda+(k+1)\sigma}$ of H-representations, given by the H-equivariant map $V^G_{\lambda+k\sigma} \to V^G_{\lambda+(k+1)\sigma} : v \mapsto \operatorname{pr}(v \otimes v^H)$ where $v^H \in V_\sigma^G$ is non-zero and H-fixed and where pr is the Cartan projection. This implies that any H-isotypical space $(\operatorname{ind}_{H_*}^H \pi_{\lambda_*}^{H_*})(\mu)$ occurs in $V^G_{\lambda+k\sigma}$ for $k \gg 0$. $\square$

An alternative proof of Theorem 3.1 was communicated to the author by Michel Brion. The proof goes along the following lines. Let $B \subset G$ and $B_H \subset H$ be Borel subgroups of G and H, respectively, for which $BH \subset G$ is open and let $B = TU$ and $B_H = T_H U_U$ be Levi decompositions. Let U_H^- be opposite to B_H in H. The torus $T \times T_H$ acts on $G/U \times H/U_H^-$ from the right, and H acts diagonally on this space on the left. We have

$$\mathbb{C}[G/U \times H/U_H^-]^{(H\times T\times T_H)}_{(0,\lambda,\mu)} = \operatorname{Hom}_H(V_\lambda^G, V_\mu^H).$$

Let $0 \neq s \in \mathbb{C}[G]^{(H\times B)}_{(0,\sigma)}$. Then $\mathbb{C}[HB/U]_{(s)} = \mathbb{C}[G][s^{-1}]$ is naturally graded. Viewing s as a regular function on $G/U \times H/U_H^-$ yields, after localizing in the ideal $(s) \subset \mathbb{C}[G/U \times H/U_H^-]$,

$$\mathbb{C}[HB/U \times H/U_H^-]^{(H\times T\times T_H)}_{(0,\lambda,\mu)} = \bigcup_{n\geq 0} \mathrm{Hom}_H(V^G_{\lambda+n\sigma}, V^H_\mu).$$

On the other hand, $HB/U = HLU/U = HL/U_L$, where $U_L = B \cap L$. This can be seen as follows: According to the local structure theory, the multiplication map $Q^u \times LH \to QH$ is an isomorphism. This holds also true for $Q^u \times U_L \to U$. We have

$$HQ/U = (HL \times Q^u)/(Q^u \times U_L) = HL \times^{U_L} (Q^u/Q^u),$$

which is isomorphic to HL/U_L. In turn, $HL/U_L = H \times^{H_*} (L/U_L)$, from which we deduce

$$\mathbb{C}[HB/U \times H/U_H^-]^{(H\times T\times T_H)}_{(0,\lambda,\mu)} = \mathrm{Hom}_{H_*}(V^L_\lambda, V^H_\mu).$$

Since V^L_λ is an irreducible H_*-module of highest weight $\lambda_* = \lambda|_{B_{H_*}}$, we find

$$\mathrm{Hom}_{H_*}(V^{H_*}_{\lambda_*}, V^H_\mu) = \bigcup_{n\geq 0} \mathrm{Hom}_H(V^G_{\lambda+n\sigma}, V^H_\mu),$$

from which the result follows.

Remark 3.2. It follows that $P^+_G(\mu)$ has *finite behavior*, i.e. the set $P^+_G(\mu)$ projects onto the finite set $P^+_{H_*}(\mu)$, and an *asymptotic behavior*, i.e. we can add elements of $P^+_G(0)$ without leaving the set $P^+_G(\mu)$. In the multiplicity free cases where (G, H) is of rank one, the various μ-wells have been determined in [19]. In these cases, there exists for each $\nu \in P^+_{H_*}(\mu)$ a minimal element $\lambda_\nu \in P^+_G(\mu)$ that projects onto ν. The set of these λ_νs is denoted by $B(\mu)$, which we call the bottom of the μ-well. In the rank one cases $B(\mu) \to P^+_{H_*}(\mu)$ is a bijection.

For general $\mu \in P^+_H$ there need not be a minimal element in $P^+_G(\mu)$ over all $\tau \in P^+_{H_*}(\mu)$, see e.g. [9, Remark 3.1]. In the multiplicity free case that is studied in this paper it may still be the case that such a minimal element exists. In this case the μ-well would have a bottom $B(\mu) \subset P^+_G(\mu)$ such that $P^+_G(\mu) = B(\mu) + P^+_G(0)$. This is the case for the three examples of higher rank that are discussed in the next section.

Remark 3.3. It is interesting whether all the multiplicity free branchings from H to H_* can be described as lattice points of a convex polytope. The branching of G_2 to SO_4 does not have this property: lattice points are missing on the boundary of the convex polytope that contains all SO_4-types in the restriction of an irreducible G_2-module of highest weight $k\varpi_2$ (the shorter fundamental weight). However, this is not a counterexample, for this restriction is not multiplicity free.

4. Classification II

To finish the proof of the classification, we need a number of results.

Lemma 4.1. *Let $H = \mathrm{GL}_{n-2} \times \mathrm{SL}_2 \subset \mathrm{SL}_n$, $P_1 = P_{\{\alpha_i\}^c}$, $P_2 = B_{\mathrm{SL}_2}$ and $P = P_1 \times P_2$. If $k, \ell \in \mathbb{N}$ and $k, \ell \geq 1$, then the G-representation $\mathrm{ind}^G_P(k\omega_i + \ell\omega')$ is not multiplicity free.*

We know that there exist $k', \ell' \in \mathbb{N}_{\geq 1}$ such that $\mathrm{ind}^G_P(k'\omega_i + \ell'\omega')$ is not multiplicity free as a G-representation, since $P \subset G$ is not spherical by [16]. However, it is *a priori* not clear that this holds for all pairs (k', ℓ') different from $(0, 0)$.

Proof of Lemma 4.1. By Theorem 3.1 it is sufficient to check that $\pi^H_{k\omega_i+\ell\omega'}$ restricted to H_* is not multiplicity free. The center of H_* is isomorphic to $\mathbb{C}^\times\times\mathbb{C}^\times$, its semi-simple part is isomorphic to SL_{n-4}. Fix an SL_{n-4}-module V that occurs in $\pi^{\mathrm{GL}_{n2}}_{k\omega_i}$ with multiplicity $k+1$. The $k+1$ copies are denoted by $V^p, p=0,\dots,k$.

The restriction of $\pi^H_{k\omega_i+\ell\omega'}$ to SL_{n-4} contains V with multiplicity $(\ell+1)(k+1)$. Indeed, for any weight vector $x^{\ell-j}y^j$ of $\pi^{\mathrm{SL}_2}_{\ell\omega'}$ there is a copy $V^p_{\ell-j,j}$ of V. The center of H_* cannot separate all these copies. In fact, there are constants $c_1,c_2\in\mathbb{Z}$ such that $(z_1,z_2)\in\mathbb{C}^\times\times\mathbb{C}^\times$ acts on $V^p_{\ell-j},j$ via $z^{c_1-(n-4)(p-j)}z^{c_2+(n-4)(p+\ell-j)}$. Hence, the H_*-modules $V^0_{\ell,0}$ and $V^1_{\ell-1,1}$ are indistinguishable. □

Lemma 4.2. *Let $H=\mathrm{SL}_{n-2}\times\mathrm{SL}_2\subset\mathrm{SL}_n$, $P_1=P_{\{\alpha_i\}^c}$, $P_2=B_{\mathrm{SL}_2}$ and $P=P_1\times P_2$. If $k,\ell\in\mathbb{N}$ and $k,\ell\geq 1$, then the G-representation $\mathrm{ind}^G_P(k\omega_i+\ell\omega')$ is not multiplicity free.*

Proof. The proof is similar to the proof of Lemma 4.1. The SL_{n-4} modules are now indexed as $V^p_{\ell-2j}$ and the center of H_*, which is isomorphic to $\mathbb{C}^\times$, acts on these modules via $z\mapsto z^{c+2p+\ell-2j}$. As $0\leq p\leq k$ and $0\leq j\leq\ell$, the values $(k,j)=(0,0)$ and $(1,1)$ yield different, but isomorphic H_*-modules. □

Lemma 4.3. *Let $(G,H)=(\mathrm{Sp}_{2n},\mathrm{Sp}_{2n-2}\times\mathrm{Sp}_2)$ and $\ell\in\mathbb{N}$. Then*

$$P^+_G(\omega_i+\ell\omega_n)\cap P^+_G(\omega_i+(\ell+2)\omega_n)\neq\emptyset.$$

Proof. In view of Theorem 3.1, it is sufficient to show that $P^+_{H_*}(\omega_i+\ell\omega_n)\cap P^+_{H_*}(\omega_i+(\ell+2)\omega_n)\neq\emptyset$, because $\lambda_{\mathrm{sph}}=\varpi_2$ is indivisible. Consider the subgroup $H_1=\mathrm{Sp}_2\times\mathrm{Sp}_{2n-4}\times\mathrm{Sp}_2\subset H$ whose embedding is standard, i.e. the embedding of the first two factors is similar to $H\subset G$ and the third factor is equality. The general isotropy group $H_*=\mathrm{Sp}_{2n-4}\times\mathrm{Sp}_2$ is embedded in H_1 by the diagonal embedding of the factor Sp_2. The restriction of H-types in a two-dimensional face to H_1 is described in [37, Prop. 5.1]. The restriction of $\pi^H_{\omega_i+\ell\omega_n}$ contains H_1 types of highest weight $\ell'w_1+\kappa+\ell w_n$ with $\ell'\neq 0$, where κ is some Sp_{2n-4}-weight and w_1 and w_n are the fundamental weights of the Sp_2-factors. Restricting this representation to H_* amounts to calculating the restriction of $\pi^{\mathrm{Sp}_2}_{\ell'}\otimes\pi^{\mathrm{Sp}_2}_{\ell(+2)}$ to the diagonal. This restriction is multiplicity free and the Sp_2-representations of highest weight that occur are $\{|\ell(+2)-\ell'|,|\ell(+2)-\ell'|+2,\dots,\ell(+2)+\ell'\}$. In particular, $\ell+\ell'$ occurs for both the cases ℓ and $\ell+2$. □

Remark 4.4. In verifying the final items for the proof of Theorem 2.1, we will use the following results and notations.

1. The pair $(G,H)=(\mathrm{SL}_2\times\mathrm{SL}_2,\mathrm{SL}_2)$, with SL_2 embedded diagonally in G is spherical. A Borel subgroup of H remains spherical in G. The Clebsch-Gordan branching rules yield $P^+_G(\ell\omega)=\{m\varpi+n\varpi'':|m-n|\leq\ell\leq m+n \text{ and } m+n\equiv\ell \pmod 2\}$.
2. The pair $(G,H)=(\mathrm{SL}_2^3,\mathrm{SL}_2)$, with SL_2 embedded diagonally in G, is a spherical pair, but a Borel subgroup of H is no longer spherical in G, by dimension count. An application of the Clebsch-Gordan branching rules yields $P^+_G(0)=\{\ell_1\omega+\ell_2\omega'+\ell_3\omega'':$ $\ell_2-\ell_3|\leq\ell_1\leq\ell_2+\ell_3 \text{ and } \ell_2+\ell_3\equiv\ell_1 \pmod 2\}$.
3. If $(G,H)=(\mathrm{Sp}_{2m},\mathrm{Sp}_{2m-2}\times\mathrm{Sp}_2)$, then $P^+_G(\ell\omega')=\ell\varpi_1+\mathbb{N}\varpi_2$, as follows from [19, Thm. 5.2].
4. If $(G,H)=(\mathrm{SL}_n,\mathrm{SL}_{n-2}\times\mathrm{SL}_2)$ or $(G,H)=(\mathrm{SL}_n,\mathrm{GL}_{n-2}\times\mathrm{SL}_2)$, then $P^+_G(m\omega')\cap P^+_G(n\omega')=\emptyset$ if $m\neq n$. This follows from the branching rule from H to H_* and Theorem 3.1. Indeed, the restriction $\pi^H_{m\omega'}|_{H_*}$ remains irreducible.

(5) The irreducible representation of $H = H_1 \times \cdots \times H_k$ of highest weight $\ell\omega + \cdots + \ell^{(k)}\omega^{(k)}$ is denoted by $\pi_{\ell\omega} \otimes \cdots \otimes \pi^{(k)}_{\ell^{(k)}\omega^{(k)}}$.

Proof of Theorem 2.1. The only items that still need verification are № 15, 17–20. In all cases we obtain a list of candidates by projecting onto the simple factors of G.

(№ 15 and 17). In both cases, the candidates are $\{\alpha_i, \alpha', \alpha''_j\}$ and its subsets. In both cases, $\{\alpha_i\}$ is discarded by Lemmas 4.1 and 4.2. Indeed, the induction of $\pi_{\ell\omega_i} \otimes \pi'_0 \otimes \pi''_0$ to the intermediate subgroup $\mathrm{GL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$ or $\mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$ yields representations $\pi_{\ell\omega_i} \otimes \pi'_{\ell'\omega} \otimes \pi'_{\ell'\omega'} \otimes \pi''_0$, with $\ell \in \mathbb{N}$. But the induction of $\pi_{\ell\omega_i} \otimes \pi'_{\ell'\omega}$ to SL_n is not multiplicity free.

The set $\{\alpha', \alpha''_j\}$ is excluded by Lemma 4.3. Indeed, inducing a representation $\pi_0 \otimes \pi'_{\ell'} \otimes \pi''_{\ell''\omega''_j}$ to $\mathrm{GL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$ or $\mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$ gives representations $\pi_0 \otimes \pi'_{2\omega'} \otimes \pi'_{\ell'\omega'} \otimes \pi''_{\ell''\omega''_j}$ and $\pi_0 \otimes \pi'_{2\omega'} \otimes \pi'_{(\ell'+2)\omega'} \otimes \pi''_{\ell''\omega''_j}$.

The set $\{\alpha'\}$ does give a parabolic subgroup of H that remains spherical in G. Indeed, inducing the representation $\pi_0 \otimes \pi'_{\ell\omega'} \otimes \pi''_0$ to the intermediate subgroup $\mathrm{GL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$ or $\mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$ gives representations $\pi_0 \otimes \pi'_{m\omega'} \otimes \pi'_{n\omega'} \otimes \pi''_0$, with $|m-n| \le \ell \le m+n$ and $m+n \equiv \ell \pmod 2$. It could still be that the various $P^+_{\mathrm{SL}_n}(m\omega')$ have non-empty intersection, but these representations are separated by the second tensor, in view of Remark 4.4(3).

The set $\{\alpha''_j\}$ also yields a parabolic subgroup of H that remains spherical in G. Indeed, inducing a representation $\pi_0 \otimes \pi'_0 \otimes \pi''_{\ell''\omega_j}$ to the intermediate subgroup $\mathrm{GL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$ or $\mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$ gives representations $\pi_0 \otimes \pi'_{\ell'\omega'} \otimes \pi'_{\ell'\omega'} \otimes \pi''_{\ell''\omega_j}$. Possible higher multiplicities of the second factor of the induction to G are separated by the first factor, in view of Remark 4.4(4).

(№ 18). Any subset is discarded. Indeed the induction of any H-representation other than the trivial one, is not multiplicity free, in view of of Remark 4.4(1,2) in combination with Lemma 4.3.

(№ 19). The subsets $\{\alpha_i, \alpha'\}, \{\alpha', \alpha''_j\}$ and $\{\alpha_i, \alpha''_j\}$ are discarded, in view of of Remark 4.4(1) in combination with Lemma 4.3.

The sets $\{\alpha'\}, \{\alpha_i\}$ and $\{\alpha''_j\}$ do yield a parabolic subgroup of H that remains spherical in G, by Remark 4.4(3).

(№ 20). The subsets $\{\alpha\}, \{\alpha'\}, \{\alpha''\}$ and $\{\alpha'''\}$ are all discarded in view of Lemma 4.3. □

5. Examples

In this section we determine the μ-wells of three multiplicity free induced modules. We show that each μ-well that we describe has a partial ordering with good behavior with respect to branching. This ordering is used in Section 6 to construct families orthogonal polynomials in several variables that have the Sturm-Liouville-property.

5.1. **The case** $(\mathrm{Spin}_9, \mathrm{Spin}_7)$**.** Let $(G,H) = (\mathrm{Spin}_9, \mathrm{Spin}_7)$. The embedding of H in G is as follows. Let $G_1 = \mathrm{Spin}_8 \subset G$ be the standard embedding. Let $\tau : \mathfrak{so}_8 \to \mathfrak{so}_8$ be the outer automorphism interchanging the roots α_1 and α_3. We denote the corresponding Lie group automorphism by $\tilde{\tau} : \mathrm{Spin}_8 \to \mathrm{Spin}_8$. Let $H \subset G_1$ be the standard embedding. The embedding that we take is $\tilde{\tau} : H \to G$.

Lemma 5.1. *Let $\nu = b_1\epsilon_1 + b_2\epsilon_2 + b_3\epsilon_3 + b_4\epsilon_4$ be a highest weight of G_1 and $k \in \mathbb{N}$. Then the restriction $\pi^{G_1}_\nu|_H$ contains $\pi^H_{k\epsilon_1}$ if and only if*

$$b_1 + b_3 \geq k \geq b_1 + b_4 \geq 0, \quad b_1 = b_2, \quad b_3 = -b_4. \tag{2}$$

Proof. The representation $\pi^H_{k\epsilon_1}$ occurs in the restriction of $\pi^{G_1}_\nu$ if and only if $\pi^{\mathrm{Spin}_7}_{\tau(k\epsilon_1)}$ occurs in the restriction of $\pi^{G_1}_\nu$ to $\mathrm{Spin}_7 \subset \mathrm{Spin}_8$, the standard embedding. This is equivalent to the usual interlacing conditions for $k\epsilon_1$ and $\tau(\nu)$, that describe the restriction of Spin_8 to Spin_7. The matrix of τ, restricted to the standard Cartan subalgebra, is given by

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

with respect to the basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. The result follows from some straightforward manipulations. □

Proposition 5.2. *The restriction to H of an irreducible G-representation of highest weight $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\epsilon_4 \in P^+_G$ contains $\pi^H_{k\epsilon_1}$ if and only if $a_2 + a_4 \geq k \geq a_2 - a_4$.*

Proof. We call an irreducible G_1-representation *useful* if its restriction to H contains $\pi^H_{k\epsilon_1}$. The only possible useful G_1-module that occurs in the restriction of π^G_λ is of highest weight $\nu = a_2\epsilon_1 + a_2\epsilon_2 + a_4\epsilon_3 - a_4\epsilon_4$. This is representation is precisely useful, if $a_2 + a_4 \geq k \geq a_2 - a_4$, as desired. □

Theorem 5.3. *Let $\mu = k\epsilon_1$. Define $B(\mu) = \{s\varpi_2 + t\varpi_3 + (k-s-t)\varpi_4 : s, t \in \mathbb{N}, s+t \leq k\}$. An element $\nu \in B(\mu)$ is written as $\nu = \nu(k; s, t) = s\varpi_2 + t\varpi_3 + (k-s-t)\varpi_4$. The map*

$$\Lambda_k : \mathbb{N} \times \mathbb{N} \times B(\mu) \to P^+_G(\mu) : (d_1, d_2, \nu) \mapsto \nu + d_1\varpi_1 + d_2\varpi_4$$

is a bijection of sets.

Proof. The fundamental spherical weights for (G, H) are ϖ_1 and ϖ_4. The set $P^+_G(\mu)$ is closed under addition by elements $d_1\varpi_1 + d_2\varpi_4$, $d_1, d_2 \in \mathbb{N}$.

Let $\lambda \in P^+_G(\mu)$ with $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\epsilon_4$. The element $\lambda - (a_1 - a_2)\varpi_1 - (a_2 + a_4 - k)\varpi_4$ lies in $B(\mu)$. On the other hand, given an element $\nu \in B(\mu)$, we can write it as $\nu = c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 + c_4\epsilon_4$ with $c_2 + c_4 = k$ and $c_1 = c_2$, so we cannot subtract $d_1\varpi_1 + d_2\varpi_4$, without leaving $P^+_G(\mu)$. This shows that Λ_k is an bijection. □

Definition 5.4. *Let $d : P^+_G(k\epsilon_1) \to \mathbb{N} : \lambda = \Lambda_k(d_1, d_2, \nu(s,t)) \mapsto d_1 + d_2$. Let $\preceq_{k\epsilon_1}$ be the partial ordering on $P^+_G(k\epsilon_1)$ defined by*

$$\lambda \preceq_{k\epsilon_1} \lambda' \Leftrightarrow (d(\lambda) < d(\lambda') \text{ or } d(\lambda) = d(\lambda') \text{ and } s + t \leq s' + t').$$

Proposition 5.5. *Let $\lambda \in P^+_G(k\epsilon_1)$ and $\sigma \in \{\varpi_1, \varpi_4\}$. Let $\pi^G_{\lambda'}$ be an irreducible representation that occurs in the decomposition of $\pi^G_\lambda \otimes \pi^G_\sigma$. If $\lambda' \in P^+_G(k\epsilon_1)$, then $\lambda' \preceq_{k\epsilon_1} \lambda + \sigma$.*

Proof. The highest weights of the irreducible constituents in the decomposition of $\pi^G_\lambda \otimes \pi^G_\sigma$ are contained in the set $\{\lambda + \xi : \xi \text{ a weight of } V^G_\sigma\}$. We collect the coefficients of the weights $\xi = x_1(\varpi_2 - \varpi_4) + x_2(\varpi_3 - \varpi_4) + x_3\varpi_1 + x_4\varpi_4$ of π^G_σ in the matrices M_σ below.

$$M_{\varpi_1} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix},$$

$$M_{\varpi_4} = \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

We observe that $x_3 + x_4 \leq 1$ and if $x_3 + x_4 = 1$, then $x_1 + x_2 \leq 0$, which establishes the claim. $\square$

Remark 5.6. There is also a twisted embedding $H = \mathrm{Spin}_7 \to \mathrm{SO}_9 = G'$. In this case the fundamental spherical weights are $\{\varpi_1, 2\varpi_4\}$. We can also determine the sets $P^+_{G'}(k\epsilon_1)$ and $B(k\epsilon_1)$, but the bottom $B(k\epsilon_1)$ is no longer contained in an affine space. In fact, it is contained in the union of two affine spaces, the one a ϖ_4-translate of the other. It is unclear how to define a partial ordering on $P^+_{G'}(k\epsilon_1)$. Moreover, as $\dim V^{G'}_{2\varpi_4} = 126$, it would also be cumbersome to check such an ordering for nice properties like that of Proposition 5.5.

5.2. **The case** $(\mathrm{SL}_3 \times \mathrm{SL}_3, \mathrm{diag}(\mathrm{SL}_3))$**.** Let $H = \mathrm{SL}_3$ let $B \subset H$ be the Borel subgroup consisting of upper triangular matrices and let $T \subset B$ be the torus consisting of the diagonal elements. Let $P \subset H$ be the parabolic subgroup corresponding to the set $\{\alpha_1\}^c$. Let $G = H \times H$ and let $\mathrm{diag}(H)$ be the diagonal, which we identify with H. Then (G, H, P) is a multiplicity free system with (G, H) a symmetric pair with involution $\theta(x, y) = (y, x)$. The image of $T \to G : t \mapsto (t, t^{-1})$ is denoted by A. We identify $Z_H(A) = \mathrm{diag}(T)$ with T. Let B^- denote the Borel subgroup opposite to B with $B \cap B^- = T$. The Borel subgroup $B \times B^-$ determines a notion of positivity for roots and weights of G. The fundamental weights of H are denoted by ω_i, those of G by $(\omega_i, 0)$ and $(0, -\omega_i)$. The spherical weights are $(\omega_i, -\omega_i)$. Fix $k \in \mathbb{N}$. Then

$$P^+_T(k\omega_1) = \{k\omega_1 - s\alpha_1 - t\alpha_3 | s, t \in \mathbb{N} \text{ and } s + t \leq k\},$$

which follows from the ordinary branching rules of H to T. Define $\widetilde{\alpha}_1 = (\omega_1 - \omega_2, \omega_1), \widetilde{\alpha}_2 = (\omega_2, \omega_2 - \omega_1)$ and $\widetilde{\alpha}_3 = (\omega_1, \omega_2)$.

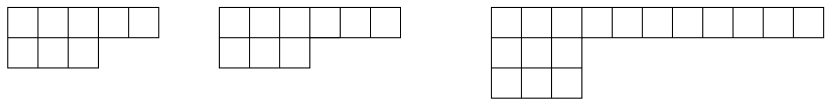
Theorem 5.7. *Define* $B(k\omega_1) = \{(k\omega_1, 0) - s\widetilde{\alpha}_1 - t\widetilde{\alpha}_3 | s, t \in \mathbb{N}$ *and* $s + t \leq k\}$. *Then*

$$\Lambda_k : \mathbb{N} \times \mathbb{N} \times B(k\epsilon_1) \to P^+_G(\mu) : (d_1, d_2, \nu) \mapsto d_1(\omega_1, -\omega_1) + d_2(\omega_2, -\omega_2) + \nu$$

is a bijection of sets.

Proof. The orthogonal projection of $P^+_G(k\omega_1)$ along the span of $(\omega_1, -\omega_1)$ and $(\omega_2, -\omega_2)$ maps onto the image of $P^+_T(k\omega_1) \to P_G : \omega_i \mapsto \frac{1}{2}(\omega_i, \omega_i)$. Since the fundamental spherical weights are indivisible, it is sufficient to show (1) $B(k\omega_1) \subset P^+_G(k\omega_1)$ and (2) if $\nu \in B(k\omega_1)$ and $\sigma \in P^+_G(0) \backslash \{0\}$, then $\nu - \sigma \notin P^+_G(k\omega_i)$.

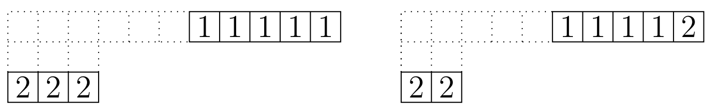
Let $\nu = \nu(k; s, t) := (k\omega_1, 0) - s\widetilde{\alpha}_1 - t\widetilde{\alpha}_3 \in B(\mu)$. Then $\nu = ((k-s-t)\omega_1 + s\omega_2, -s\omega_1 - t\omega_2)$. Write $\mu_1 = (k-s-t)\omega_1 + s\omega_2$. The irreducible H-representation of lowest weight $-s\omega_1 - t\omega_2$ is of highest weight $\mu_2 = t\omega_1 + s\omega_2$. To prove (1) we have to show that $\pi^H_{k\omega_1}$ occurs in the decomposition of the H-representation $\pi^H_{\mu_1} \otimes \pi^H_{\mu_2}$. We will use the Littlewood-Richardson rule to prove (1) and (2). For a reference for this branching rule, and the notions we use below, see e.g. [14, §9.3.5]. We identify $\mu_1 = [k - t, s]$ and $\mu_2 = [s + t, s]$ with their Young diagrams, see Figure 2.



$\mu_1 = [k-t, s]$ $\quad$ $\mu_2 = [s+t, s]$ $\quad$ $\lambda = [k+s, s, s]$

FIGURE 2. A Young diagram $[a, b, c]$ has a boxes on the first row, b on the second and c on the third. In this picture, $k = 8$ and $s = t = 3$.

We enlarge the Young diagram $k\omega_1 = [k]$ to $\lambda = [k+s, s, s]$. This is equivalent to replacing $\pi^H_{k\omega_1}$ by $\pi^H_{k\omega_1} \otimes \det^s$, but this is harmless because we work on SL_3. The Littlewood-Richardson rule says: $\pi^H_{\mu_1} \otimes \pi^H_{\mu_2}$ contains $\pi^H_{k\omega_1}$ if and only if there is a Littlewood-Richardson skew tableau λ/μ_2 of weight μ_1. In Figure 3 we have drawn the desired diagram, thereby proving (1).



λ/μ_1 of weight μ_1 $\quad$ $\widetilde{\lambda}/\widetilde{\mu}_1$ of weight $\widetilde{\mu}_1$

FIGURE 3

Let $\nu = \nu(k; s, t) \in B(k\omega_1)$. The tensor product that corresponds to the element $\nu - (\omega_1, -\omega_1)$ is identified with the tensor product $\widetilde{\mu}_1 \otimes \widetilde{\mu}_2$, where $\widetilde{\mu}_1 = [k-t-1, s], \widetilde{\mu}_2 = [s+t-1, s-1]$. The Young tableau of $k\omega_1$ is replaced by $\widetilde{\lambda} = [s-1+k, s-1, s-1]$. In Figure 3 we have depicted the only possible semistandard skew tableau $\widetilde{\lambda}/\widetilde{\mu}_2$ of weight $\widetilde{\mu}_1$. However, this is not a Littlewood-Richardson skew tableau. Hence $\nu - (\omega_1, -\omega_1) \notin P^+_G(k\omega_1)$. Similarly one shows that $\nu - (\omega_2, -\omega_2) \notin P^+_G(k\omega_1)$. This proves (2). $\square$

Definition 5.8. *Let $d : P^+_G(k\omega_1) \to \mathbb{N} : \lambda = \Lambda_k(d_1, d_2, \nu(s,t)) \mapsto d_1 + d_2$. Let $\preceq_{k\omega_1}$ be the partial ordering on $P^+_G(k\omega_1)$ defined by*

$$\lambda \preceq_{k\omega_1} \lambda' \Leftrightarrow (d(\lambda) < d(\lambda') \text{ or } d(\lambda) = d(\lambda') \text{ and } s + t \geq s' + t')\,.$$

Proposition 5.9. *Let $\lambda \in P^+_G(k\omega_1)$ and $\sigma \in \{(\omega_1, -\omega_1), (\omega_2, -\omega_2)\}$. Let $\pi^G_{\lambda'}$ be an irreducible representation that occurs in the decomposition of $\pi^G_\lambda \otimes \pi^G_\sigma$. If $\lambda' \in P^+_G(k\omega_1)$, then $\lambda' \preceq_{k\omega_1} \lambda + \sigma$.*

Proof. The proof is similar to the proof of Proposition 5.5. We collect the weights of the fundamental representation $(\omega_1, -\omega_1)$ as columns of the matrix $M_{(\omega_1,-\omega_1)}$, where the weights are expressed in the basis $\{\widetilde{\alpha}_1, \widetilde{\alpha}_2, (\omega_1, -\omega_1), (\omega_2, -\omega_2)\}$.

$$M_{(\omega_1,-\omega_1)} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

We observe that $x_3 + x_4 \leq 1$ and if $x_3 + x_4 = 1$, then $x_1 + x_2 \geq 0$. For the case $(\omega_2, -\omega_2)$, the columns of $M_{(\omega_2,-\omega_2)}$ are a permutation of the columns of $-M_{(\omega_1,-\omega_1)}$ and we can make the same observations. This establishes the claim. $\square$

Remark 5.10. Most of the analysis in this subsection carries over *mutatis mutandis* to the case where $H = \mathrm{SL}_{n+1}$, except for Proposition 5.9, which soon becomes involved for large n.

5.3. **The case** $(\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n}, \mathrm{Sp}_{2m-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$**.** Let $(G, H) = (\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n}, \mathrm{Sp}_{2m-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$. Using Remark 4.4(1,3) and induction via the intermediate subgroup $\mathrm{Sp}_{2m-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$, we obtain

$$P_G^+(0) = \mathbb{N}(\varpi_1, \varpi_1') \oplus \mathbb{N}(\varpi_2, 0) \oplus \mathbb{N}(0, \varpi_2').$$

Restricting the H-representation of highest weight $\mu = (0, \ell\omega', 0)$ to H_* is really the restriction of an irreducible Sp_2-representation to its maximal torus and it decomposes into $\ell + 1$ (different) one-dimensional weight spaces.

Theorem 5.11. *Define* $B(k\omega') = \{(\ell_1\varpi_1, \ell_2\varpi_1') : \ell_1 + \ell_2 = \ell\}$. *Then*

$$\Lambda_k : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times B(k\omega') \to P_G^+(k\omega'),$$
$$(d_1, d_2, d_3, \nu) \mapsto d_1(\varpi_1, \varpi_1') + d_2(\varpi_2, 0) + d_3(0, \varpi_2') + \nu$$

is a bijection of sets.

Proof. Using Remark 4.4(1,3), we see that $B(k\omega') \subset P_G^+(k\omega')$. Subtracting $(\varpi_2, 0)$ or $(0, \varpi_2')$ from an element in $B(k\omega')$ gives an element outside P_G^+. Subtracting $r(\varpi_1, \varpi_1')$ yields $(p_1\varpi_1, p_2\varpi_1')$ with $p_1 + p_2 = \ell - 2r$, whence $(p_1\varpi_1, p_2\varpi_1') \notin P_G^+(\mu)$. ☐

Definition 5.12. *Let* $d : P_G^+(k\omega') \to \mathbb{N} : \lambda = \Lambda_k(d_1, d_2, d_3, \nu(\ell_1, \ell_2)) \mapsto d_1 + d_2 + d_3$. *Let* $\preceq_{k\omega'}$ *be the partial ordering on* $P_G^+(k\omega_1)$ *defined by*

$$\lambda \preceq_{k\omega'} \lambda' \Leftrightarrow (d(\lambda) < d(\lambda') \text{ or } d(\lambda) = d(\lambda') \text{ and } d_1 \le d_1').$$

Proposition 5.13. *Let* $\lambda \in P_G^+(k\omega_1)$ *and* $\sigma \in \{(\varpi_1, \varpi_1'), (\varpi_2, 0), (0, \varpi_2')\}$. *Let* $\pi_{\lambda'}^G$ *be an irreducible representation that occurs in the decomposition of* $\pi_\lambda^G \otimes \pi_\sigma^G$. *If* $\lambda' \in P_G^+(k\omega')$, *then* $\lambda' \preceq_{k\omega'} \lambda + \sigma$.

Proof. Write $\nu = \varpi_1 - \varpi_1', \sigma_1 = \varpi_1 + \varpi_1', \sigma_2 = \varpi_2$ and $\sigma_3 = \varpi_2'$. The only weights of $V_{\sigma_2}^G$ that are in $\mathbb{Z}\nu + \mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_3$ are $\pm\sigma_2$ and 0. Similarly for $V_{\sigma_3}^G$. This shows that $d(\lambda) = d(\lambda')$ implies $\lambda = \lambda'$, in case $\sigma \in \{\sigma_2, \sigma_3\}$. The weights of $V_{\sigma_1}^G$ that occur in $\mathbb{Z}\nu + \mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_3$ are listed in Table 4.

1	$\pm(\epsilon_1 + \epsilon_1')$	$\pm\sigma_1$	2	$\pm(\epsilon_1 - \epsilon_1')$	$\pm\nu$
3	$\pm(\epsilon_1 + \epsilon_2')$	$\pm(\nu + \sigma_3)$	4	$\pm(\epsilon_1 - \epsilon_2')$	$\pm(\sigma_1 - \sigma_3)$
5	$\pm(\epsilon_2 + \epsilon_1')$	$\pm(\sigma_2 - \nu)$	6	$\pm(\epsilon_2 - \epsilon_1')$	$\pm(\sigma_2 - \sigma_1)$
7	$\pm(\epsilon_2 + \epsilon_2')$	$\pm(\sigma_2 + \sigma_3 - \sigma_1)$	8	$\pm(\epsilon_2 - \epsilon_2')$	$\pm(\sigma_2 - \sigma_3 - \nu)$

Table 4

We observe that either the total degree (i.e. the sum of the coefficients of $\sigma_1, \sigma_2, \sigma_3$) is ≤ 0 or equal to one, in which case the coefficient of σ_1 is equal to one (item 1) or strictly less than one (items 3, 5, 7). ☐

6. Orthogonal polynomials

Let (G,H,P) be a multiplicity free system from Section 5 or one where (G,H) is a spherical pair of rank one. Fix a character $\mu : P \to \mathbb{C}^\times$ with $\mu \in P_H^+$. The group $H \times H$ acts on $\mathbb{C}[G] \otimes \mathrm{End}(V_\mu^H)$ via $(h_1,h_2)F(g) = \pi_\mu^H(h_1)F(h^{-1}gh_2)\pi_\mu^H(h_2^{-1})$. The space of invariants of this action is denoted by

$$E^\mu = (\mathbb{C}[G] \otimes \mathrm{End}(V_\mu^H))^{H\times H}.$$

Note that E^0 is an algebra and E^μ an E^0-module. Our aim is to describe this module structure explicitly, by means of orthogonal polynomials. We will first examine the vector space structure of E^μ.

Let $\mathbb{C}[G]_\lambda$ be the space of matrix coefficients of the irreducible G-representation V_λ^G. We observe

$$E^\mu = \bigoplus_{\lambda \in P_G^+(\mu)} E_\lambda^\mu, \quad \text{where } E_\lambda^\mu := (\mathbb{C}[G]_\lambda \otimes \mathrm{End}(V_\mu^H))^{H\times H}. \tag{3}$$

Since $m_\lambda^{G,H}(\mu) = 1$, the space E_λ^μ is one dimensional. Fix H-equivariant maps

$$j_\lambda^\mu : V_\mu^H \to V_\lambda^G \text{ and } p_\mu^\lambda : V_\lambda^G \to V_\mu^H, \quad \text{with } p_\mu^\lambda \circ j_\lambda^\mu = \mathrm{Id}_{V_\mu^H}.$$

Define $\Phi_\lambda^\mu \in \mathbb{C}[G]_\lambda \otimes \mathrm{End}(V_\mu^H)$ by $\Phi_\lambda^\mu(g) = p_\mu^\lambda \circ \pi_\lambda^G(g) \circ j_\lambda^\mu$. Then $\Phi_\lambda^\mu \in E_\lambda^\mu$ and Φ_λ^μ is independent of j_λ^μ and p_μ^λ. The functions in E^μ are called spherical functions of type μ. The functions Φ_λ^μ are called elementary spherical functions of type μ, associated to λ. The spherical functions satisfy the following transformation behavior,

$$\Phi(h_1gh_2) = \pi_\mu^H(h_1) \circ \Phi(g) \circ \pi_\mu^H(h_2), \quad \text{for all } h_1,h_2 \in H \text{ and } g \in G. \tag{4}$$

If $\mu = 0$ then E^0 is an algebra, finitely generated by the zonal spherical functions $\phi_i := \Phi_{\sigma_i}^0$, where $\sigma_1,\ldots,\sigma_r$ generate the weight monoid $P_G^+(0)$. In the cases under consideration, $P_G^+(0)$ is actually freely generated. This implies that $E^0 = \mathbb{C}[\phi_1,\ldots,\phi_r]$, where $r = r_G(G/H)$. Recall that for all the cases that we consider, there is an isomorphism $\Lambda : \mathbb{N}\times\cdots\times\mathbb{N}\times B(\mu) \to P_G^+(\mu)$. An element $(d_1,\ldots,d_r) \in \mathbb{N}^r$ is denoted by d. We define the degree of the spherical function $\Phi_{\Lambda(d,\nu)}^\mu$ to be $|d| = d_1 + \cdots + d_r$.

Fix maximal compact subgroups $G_0 \subset G$ and $H_0 \subset H$ with $H_0 \subset G_0$. Define the pairing

$$\langle \Phi_1, \Phi_2 \rangle_{\mu,G_0} = \int_{G_0} \mathrm{tr}\left(\Phi_1(g)^* \Phi_2(g)\right) dg, \quad \Phi_1,\Phi_2 \in E^\mu,$$

with dg the normalized Haar measure on G_0. Schur orthogonality and the Peter–Weyl Theorem imply:

- The pairing $\langle\cdot,\cdot\rangle_{\mu,G_0} : E^\mu \times E^\mu \to \mathbb{C}$ is a Hermitian inner product and

$$\langle \Phi_\lambda^\mu, \Phi_{\lambda'}^\mu \rangle_{\mu,G} = c_\lambda \delta_{\lambda,\lambda'}, \quad c_\lambda = \dim(\mu)^2/\dim(\lambda).$$

- $\{\Phi_\lambda^\mu : \lambda \in P_G^+(\mu)\}$ is an orthogonal basis of E^μ.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $U(\mathfrak{g})^H$ denote the algebra of differential operators that are $\mathrm{Ad}(H)$-invariant. Let $I(\mu) \subset U(\mathfrak{h})$ be the kernel of the representation $U(\mathfrak{h}) \to \mathrm{End}(V_\mu)$ and define

$$\mathbb{D}(\mu) := U(\mathfrak{g})^H / \left(U(\mathfrak{g})^H \cap U(\mathfrak{g})I(\mu)\right).$$

The irreducible representations of the algebra $\mathbb{D}(\mu)$ correspond to irreducible representations of $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ have a subrepresentation of highest weight μ, see e.g. [11, Théorème 9.1.12]. The algebra $\mathbb{D}(\mu)$ is commutative, because all its finite dimensional representations are one-dimensional. Moreover, the elementary spherical functions in E^μ are simultaneous eigenfunctions for the algebra $\mathbb{D}(\mu)$. This ends our description of the vector space structure of E^μ. We proceed by investigating the E^0-module structure of E^μ.

Lemma 6.1. *Let $\Phi \in E^\mu$. Then there are elements $q_\nu \in E^0, \nu \in B(\mu)$ such that*

$$\Phi = \sum_{\nu \in B(\mu)} q_\nu \Phi^\mu_{\Lambda(0,\nu)}.$$

The q_ν are uniquely determined by Φ. In particular, the E^0-module E^μ is freely generated by $\{\Phi^\mu_{\Lambda(0,\mu)} : \nu \in B(\mu)\}$.

Proof. We prove the statement for an elementary spherical function. The general case follows from this one in view of the decomposition (3). Let $\lambda \in P^+_G(\mu)$. Then for all $i = 1, \ldots, r$, we can write

$$\phi_i \Phi^\mu_\lambda = \sum_{\lambda' \preceq_\mu \lambda + \sigma_i} c^\mu_{\lambda', \lambda+\sigma_i} \Phi^\mu_{\lambda'}, \tag{5}$$

where $c^\mu_{\lambda',\lambda+\sigma_i} \in \mathbb{C}$. The Borel-Weil Theorem implies that $c^\mu_{\lambda+\sigma_i,\lambda+\sigma_i} \neq 0$. Using induction on the partially ordered set $P^+_G(\mu)$, we obtain polynomials $q^\mu_{\lambda,\nu} \in E^0, \nu \in B(\mu)$, such that $\Phi^\mu_\lambda = \sum_{\nu \in B(\mu)} q^\mu_{\lambda,\nu} \Phi^\mu_{\Lambda(0,\nu)}$.

If there exist $q_\nu \in E^0$ with $\sum_\nu q_\nu \Phi^\mu_{\Lambda(0,\nu)} = 0$, then the $\Phi^\mu_{\Lambda(0,\nu)}(g) \in \mathrm{End}(V^H_\mu), \nu \in B(\mu)$ are linearly dependent for all g where $\prod_{\nu \in B(\mu)} q_\nu(g) \neq 0$.

The local structure theorem for G/H provides a parabolic subgroup $Q = Q_u \cdot L$ and a subgroup $L_0 \subset L$ containing the commutator subgroup (L, L) of L, such that the Q-equivariant map $Q \times^L L/L_0 \to G/H$ is an open embedding. The torus $A = L/L_0$ has the property, that $\psi : H \times A \times H \to G : (h, [a], h') \mapsto hah'$ is an open embedding, see e.g. [1]. Note that ψ is well defined, because $L_0 \subset H$.

It is enough to show that the functions $\psi^* \Psi^\mu_{\Lambda(0,\nu)}$ are generically linearly independent. Denote $\Psi^\mu_\lambda = (\psi^* \Phi^\mu_\lambda)|_A$. By the transformation behavior (4), it is enough to show that the restrictions of the functions $\Psi^\mu_\lambda, \nu \in B(\mu)$, are generically linearly independent. The functions Ψ^μ_λ are $\mathrm{End}_{H_*}(V^H_\mu)$-valued. As $\pi^H_\mu|_{H_*}$ decomposes multiplicity free, $\mathrm{End}_{H_*}(V^H_\mu)$ is a vector space of dimension $|B(\mu)|$.

The space $(V^G_\lambda)^{Q_u}$ is an irreducible H_*-representation, on which A acts with character $\lambda|_A$. On any other H_*-type that occurs in the decomposition V^G_λ as H_*-representation, that is stabilized by A, A acts with a character $\lambda|_A$, with $\lambda' < \lambda$. Here $\leq$ is the standard partial ordering on weights, i.e. $\lambda \geq \lambda'$ means that $\lambda - \lambda'$ is a linear combination of positive roots with non-negative integral coefficients.

If we fix a basis of $\mathrm{End}_{H_*}(V^H_\mu)$ of different highest weight vectors for H_*, then we see that the ν-entry of $\Psi^\mu_\Lambda(0,\nu)$ is linear combination of characters on A, and the coefficient of $\lambda|_A$ is non-zero. Moreover, $\lambda|_A$ is the largest character in this sum. The entries of $\Psi^\mu_{\Lambda(0,\nu)}$ of the other indexes $\nu' \in B(\mu)$ are also linear combinations of characters of A, but these are all restrictions of weights that are strictly smaller than λ.

We conclude that the $|B(\mu)| \times |B(\mu)|$ matrix valued function on A, whose columns are the $\Psi^\mu_{\Lambda(0,\nu)}, \nu \in B(\mu)$, has a determinant of positive degree. It follows that the functions $\Psi^\mu_{\Lambda(0,\nu)}, \nu \in B(\mu)$ are generically linearly independent. $\square$

It follows that E^μ is a free, finitely generated E^0-module of rank $r_{G/H}$. This ends our description of the E^0-module structure of E^μ. We proceed to construct families of matrix valued polynomials with interesting properties, which they inherit from the spherical functions. The E^0-module structure of E^μ allows us to describe the spherical functions in terms of polynomials in the fundamental spherical functions $\phi_1, \ldots, \phi_r$.

Definition 6.2. *The polynomials Q^μ_λ and Q^μ_d are defined as follows.*

- *For $\lambda = \Lambda(d, \nu') \in P^+_G(\mu)$ define $Q^\mu_\lambda(\phi) = (q^\mu_{\nu,\nu'}(\phi) : \nu \in B(\mu))$ in $\mathbb{C}^{|B(\mu)|}[\phi_1, \ldots, \phi_r]$ by*
$$\Phi^\mu_{\lambda(d,\nu')} = \sum_{\nu \in B(\mu)} q^\mu_{\nu,\nu'}(\phi_1, \ldots, \phi_r)\Phi^\mu_{\lambda(0,\nu)}.$$
- *For every multi-index $d \in \mathbb{N}^r$ define $Q^\mu_d \in \mathrm{End}(\mathbb{C}^{|B(\mu)|})[\phi_1, \ldots, \phi_r]$ as the matrix valued polynomial having the $Q^\mu_{\lambda(d,\nu')}, \nu' \in B(\mu)$ as columns.*

To simplify the notation, we fix μ and we write $\mathbb{M} = \mathrm{End}(\mathbb{C}^{|B(\mu)|})$. Moreover, we use multi-index notation, e.g. $\phi = (\phi_1, \ldots, \phi_r)$, and we drop the superscript μ in the polynomials Q_d. The polynomials Q_d are elements of $\mathbb{M}[\phi]$, which we view as a right $\mathbb{M}$-module. By $\mathbb{M}[\phi]_{|d|}$ we denote the space of all homogeneous polynomials of total degree $|d|$. It is a freely generated $\mathbb{M}$-module of rank $\binom{|d|+r-1}{|d|}$. By $\mathbb{M}[\phi]^{|d|}$ we denote the space of all polynomials of total degree $\leq |d|$.

Theorem 6.3. *The matrix valued polynomial Q_d is of total degree $|d|$. The set of matrix valued polynomials $\{Q_d : |d| \leq n\}$ is a basis of $\mathbb{M}[\phi]^n$.*

Proof. The columns of Q_d are vector valued polynomials whose entries are polynomials of total degree $|d|$, which follows from the definition of Q_d and the recurrence relation (5). The same recurrence relations imply the existence of matrices $A^d_{i,d'}, B^d_{i,d'}, C^d_{i,d'} \in \mathbb{M}$ for $i = 1, \ldots, r$ and $d' \in \mathbb{N}^r$, such that

$$\phi_i Q_d(\phi) = \sum_{|d'|=|d|+1} Q_{d'}(\phi)A^d_{i,d'} + \sum_{|d'|=|d|} Q_{d'}(\phi)B^d_{i,d'} + \sum_{|d'|=|d|-1} Q_{d'}(\phi)C^d_{i,d'}. \tag{6}$$

Let $M^n \subset \mathbb{M}[\phi]$ be the right $\mathbb{M}$-submodule generated by the set $\{Q_d : |d| \leq n\}$. By induction on n, using (6), it follows that $\mathbb{M}[\phi]^n \subset M^n$ for all $n \in \mathbb{N}$. On the other hand, $M^n \subset \mathbb{M}[\phi]^n$, whence $M^n = \mathbb{M}[\phi]^n$. Since the ranks of M^n and $\mathbb{M}[\phi]$ are the same, the set $\{Q_d : |d| \leq n\}$ is $\mathbb{M}$-linearly independent. $\square$

We will now equip $\mathbb{M}[\phi]$ with a pairing that gives a notion of orthogonality. Define $V : G \to \mathbb{M}$ by $V(g)_{\nu,\nu'} = \mathrm{tr}(\Phi_{\lambda(0,\nu)}(g)^*\Phi_{\lambda(0,\nu')}(g))$. Note that V is H-biinvariant, hence $V = W(\phi)$ with $W \in \mathbb{M}[\phi]$. The pairing

$$\langle \cdot, \cdot \rangle_W : \mathbb{M}[\phi] \times \mathbb{M}[\phi] \to \mathbb{M}, (Q, P) \mapsto \int_{G_0} Q(\phi(g))^* W(\phi(g)) P(\phi(g)) dg, \tag{7}$$

is non-degenerate and gives $\mathbb{M}[\phi]$ the structure of a right pre-Hilbert module over $\mathbb{M}$ (see [32] for more on Hilbert modules). Indeed, the pairing satisfies the following conditions: $\forall a, b \in \mathbb{C}, M \in \mathbb{M}, P, Q, R \in \mathbb{M}[\phi]$,

- $\langle P, aQ + bR\rangle_W = a\langle P, Q\rangle_W + b\langle Q, R\rangle_W$,
- $\langle P, Q \cdot M\rangle_W = \langle P, Q\rangle_W \cdot M$,
- $\langle P, Q\rangle_W^* = \langle Q, P\rangle_W$,
- $\langle P, P\rangle_W \geq 0$ with equality if and only if $P = 0$.

Here $A^* \in \mathbb{M}$ is the Hermitian adjoint of $A \in \mathbb{M}$ and $A \geq 0$ means that $A = A^*$ and A has only non-negative eigenvalues. Two elements $P, Q \in \mathbb{M}[\phi]$ are orthogonal, if $\langle P, Q\rangle_W = 0$.

Definition 6.4. *A family $\{P_d, d \in \mathbb{N}^r\} \subset \mathbb{M}[\phi]$ is called a family of matrix valued orthogonal polynomials, if (1) for all $n \in \mathbb{N}$, the set $\{P_d : |d| \leq n\}$ is a basis of the $\mathbb{M}$-submodule $\mathbb{M}[\phi]^n$, (2) for all $d, d' \in \mathbb{N}^r$, $\langle P_d, P_{d'}\rangle_W = \delta_{d,d'}C_d$ for some $C_d \in \mathbb{M}$ with $C_d > 0$.*

Proposition 6.5. *The family $\{Q_d : d \in \mathbb{N}^r\}$ is an orthogonal basis of $\mathbb{M}[\phi]$ and the matrices $\langle Q_d, Q_d\rangle_W$ are diagonal.*

Proof. Following the definitions of the polynomials Q_d and of the pairing, we see that $(\langle Q_d, Q_{d'}\rangle_W)_{\nu,\nu'} = \langle \Phi^\mu_{\Lambda(d,\nu)}, \Phi^\mu_{\Lambda(d',\nu')}\rangle_{\mu,G_0}$. This establishes all claims. ☐

Finally, we transfer the the algebra $\mathbb{D}(\mu)$ to an algebra of differential operators acting on $\mathbb{M}[\phi]$.

Definition 6.6. *Let $G_{\mu-\mathrm{reg}} \subset G$ denote the (Zariski-) open set on which the elementary spherical functions $\Phi^\mu_\Lambda(0, \nu), \nu \in B(\mu)$ are linearly independent.*

By the proof of Lemma 6.1, $G_{\mu-\mathrm{reg}}$ is non-empty. The elements of $\mathbb{D}(\mu)$ act on E^μ as differential operators. Over the open set $G_{\mu-\mathrm{reg}}$ we conjugate the elements of $\mathbb{D}(\mu)$ with the base change $\Phi^\mu_\Lambda(0, \nu), \nu \in B(\mu)$ to obtain differential operators that act on $\mathbb{M}[\phi]$. We denote the thus obtained algebra of differential operators by $\mathcal{D}(\mu)$.

Proposition 6.7. *A differential operator $D \in \mathcal{D}(\mu)$ has the functions $Q_d, d \in \mathbb{N}^r$ as simultaneous eigenfunctions, where the eigenvalues are diagonal elements $\chi(D, d) \in \mathbb{M}$ that act on the right, i.e. $DQ_d = Q_d\chi(D, d)$. Moreover, the coefficients of an element $D \in \mathcal{D}(\mu)$ are in $\mathbb{M}[\phi]$.*

Proof. Let $D \in \mathcal{D}(\mu)$. Then $D : \mathbb{M}[\phi] \to \mathbb{M}[\phi]$. All the claims, except for the last one, follow from this fact and the definitions of the $Q_d, d \in \mathbb{N}^r$ and the definition of $\mathcal{D}(\mu)$. For the last claim, note that the polynomials $\mathbb{C}[\phi]$ can be expressed as (right-) $\mathbb{M}$-linear combinations of the $Q_d, d \in \mathbb{N}^r$. Using this fact and induction on the order of D we establish the final claim. ☐

We obtain the most complete results in the symmetric case, where we have more (structure) theory available.

Example 6.8. Consider the example $(G, H) = (\mathrm{SL}_3 \times \mathrm{SL}_3, \mathrm{diag}(\mathrm{SL}_3))$, with $\mu = k\omega_1$. In this case, A is two dimensional and $(G_0, H_0) = (\mathrm{SU}(3), \mathrm{SU}(3), \mathrm{diag}(\mathrm{SU}(3)))$. Denote by $A_0 \subset A$ a maximal compact subgroup that is contained also in G_0. Furthermore, denote $M_0 = Z_{H_0}(A_0)$, which is isomorphic to a maximal compact torus in H_0.

To a differential operator $D \in \mathbb{D}(\mu)$ we can associate its radial part with respect to the decomposition $G_0 = H_0 \cdot A_0 \cdot H_0$, denoted by $\mathrm{rad}(D)$, which is a differential operator on $\mathrm{End}_M(V_\mu^K)$-valued functions A_0 with the property that $(D\Phi)|_{A_0} = \mathrm{rad}(D)(\Phi|_{A_0})$ for all smooth functions $\Phi : G_0 \to \mathrm{End}(V_\mu^H)$ with transformation behavior (4).

The functions $\Psi^\mu_{\Lambda(0,\nu)}, \nu \in B(\mu)$ are generically linearly independent. This means that we can conjugate the differential operators $\mathrm{rad}(D)$ with the base change given by the functions $\Psi^\mu_{\Lambda(0,\nu)}, \nu \in B(\mu)$. We obtain differential operators that have the functions $Q_d, d \in \mathbb{N}^2$ as eigenfunctions, where the eigenvalue is an element in $\mathbb{M}$.

The fundamental spherical functions ϕ_1, ϕ_2 give coordinates on A_0. Changing to these coordinates yields a system of matrix valued orthogonal polynomials in $\mathbb{M}[\phi_1, \phi_2]$ that is also a family of simultaneous eigenfunctions of a commutative algebra of differential operators, whose coefficients are polynomials in $\mathbb{M}[\phi_1, \phi_2]$. The matrix valued inner product can now be realized as an integration over A_0, where the measure is modified by a Jacobian factor.

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Universität Paderborn, Institut für Mathematik, Warburger Str. 100, D-33098 Paderborn

E-mail address: `vanpruijssen@math.upb.de`